AIEEE 2009 Mathematics Solutions

Code(A)

Mathematics

PART – C

61. Let a, b, c be such that $b(a+c) \neq 0$. If $\begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ (-1)^{n+2}a & (-1)^{n+1}b & (-1)^{n}c \end{vmatrix} = 0$, then the value of 'n' is

(1) zero
(2) any even integer
(3) any odd integer
(4) any integer

Sol: (3)

$$\begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^{n}\begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ a & -b & c \end{vmatrix} = \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^{n}\begin{vmatrix} a+1 & a-1 & a \\ b+1 & b-1 & -b \\ c-1 & c+1 & c \end{vmatrix}$$

$$= \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^{n+1}\begin{vmatrix} a+1 & a & a-1 \\ b+1 & -b & b-1 \\ c-1 & c & c+1 \end{vmatrix} = \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^{n+2}\begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix}$$

This is equal to zero only if n + 2 is odd i.e. n is odd integer.

62. If the mean deviation of number 1, 1 + d, 1 + 2d, ….. , 1 + 100d from their mean is 255, then the d is equal to

(1) 10.0
(2) 20.0
(3) 10.1
(4) 20.2

Sol: (3)

$$\text{Mean}(\bar{x}) = \frac{\text{sum of quantities}}{n} = \frac{\frac{n}{2}(a+l)}{n} = \frac{1}{2}[1+1+100d] = 1+50d$$

$$\text{M.D.} = \frac{1}{n}\sum|x_i - \bar{x}| \Rightarrow 255 = \frac{1}{101}[50d+49d+48d+....+d+0+d+......+50d] = \frac{2d}{101}\left[\frac{50 \times 51}{2}\right]$$

$$\Rightarrow d = \frac{255 \times 101}{50 \times 51} = 10.1$$

*63. If the roots of the equation $bx^2 + cx + a = 0$ be imaginary, then for all real values of x, the expression $3b^2x^2 + 6bcx + 2c^2$ is

(1) greater than 4ab
(2) less than 4ab
(3) greater than – 4ab
(4) less than – 4ab

Sol: (3)

$bx^2 + cx + a = 0$

Roots are imaginary $\Rightarrow c^2 - 4ab < 0 \Rightarrow c^2 < 4ab \Rightarrow -c^2 > -4ab$

$3b^2x^2 + 6bcx + 2c^2$

since $3b^2 > 0$

Given expression has minimum value

Minimum value = $\dfrac{4(3b^2)(2c^2) - 36b^2c^2}{4(3b^2)} = -\dfrac{12b^2c^2}{12b^2} = -c^2 > -4ab$.

*64. Let A and B denote the statements

A: $\cos\alpha + \cos\beta + \cos\gamma = 0$

B: $\sin\alpha + \sin\beta + \sin\gamma = 0$

If $\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -\dfrac{3}{2}$, then

(1) A is true and B is false
(2) A is false and B is true
(3) both A and B are true
(4) both A and B are false

Sol: (3)

$\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -\dfrac{3}{2}$

$\Rightarrow 2[\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta)] + 3 = 0$

$\Rightarrow 2[\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta)] + \sin^2\alpha + \cos^2\alpha + \sin^2\beta + \cos^2\beta + \sin^2\gamma + \cos^2\gamma = 0$

$\Rightarrow (\sin\alpha + \sin\beta + \sin\gamma)^2 + (\cos\alpha + \cos\beta + \cos\gamma)^2 = 0$

*65. The lines $p(p^2 + 1)x - y + q = 0$ and $(p^2 + 1)^2 x + (p^2 + 1)y + 2q = 0$ are perpendicular to a common line for

(1) no value of p
(2) exactly one value of p
(3) exactly two values of p
(4) more than two values of p

Sol: (2)

Lines must be parallel, therefore slopes are equal $\Rightarrow p(p^2 + 1) = -(p^2 + 1) \Rightarrow p = -1$

66. If A, B and C are three sets such that $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then

(1) A = B
(2) A = C
(3) B = C
(4) $A \cap B = \phi$

Sol: (3)

67. If $\vec{u}, \vec{v}, \vec{w}$ are non-coplanar vectors and p, q are real numbers, then the equality $\left[3\vec{u} \quad p\vec{v} \quad p\vec{w}\right] - \left[p\vec{v} \quad \vec{w} \quad q\vec{u}\right] - \left[2\vec{w} \quad q\vec{v} \quad q\vec{u}\right] = 0$ holds for

(1) exactly one value of (p, q)
(2) exactly two values of (p, q)
(3) more than two but not all values of (p , q)
(4) all values of (p, q)

Sol: **(1)**

$\left(3p^2 - pq + 2q^2\right)\left[\vec{u} \quad \vec{v} \quad \vec{w}\right] = 0$

But $\left[\vec{u} \quad \vec{v} \quad \vec{w}\right] \neq 0$

$3p^2 - pq + 2q^2 = 0$

$2p^2 + p^2 - pq + \left(\frac{q}{2}\right)^2 + \frac{7q^2}{4} = 0 \Rightarrow 2p^2 + \left(p - \frac{q}{2}\right)^2 + \frac{7}{4}q^2 = 0$

$\Rightarrow p = 0, q = 0, p = \frac{q}{2}$

This possible only when p = 0, q = 0 exactly one value of (p, q)

68. Let the line $\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2}$ lies in the plane $x + 3y - \alpha z + \beta = 0$. Then (α, β) equals

(1) (6, - 17)
(2) (- 6, 7)
(3) (5, - 15)
(4) (- 5, 15)

Sol: **(2)**

Dr's of line = $(3, -5, 2)$

Dr's of normal to the plane = $(1, 3, -\alpha)$

Line is perpendicular to normal $\Rightarrow 3(1)-5(3)+2(-\alpha)=0 \Rightarrow 3-15-2\alpha=0 \Rightarrow 2\alpha=-12 \Rightarrow \alpha=-6$

Also $(2, 1, -2)$ lies on the plane

$2+3+6(-2)+\beta=0 \Rightarrow \beta=7$

$\therefore (\alpha, \beta)=(-6, 7)$

*69. From 6 different novels and 3 different dictionaries, 4 novels and 1 dictionary are to be selected and arranged in a row on the shelf so that the dictionary is always in the middle. Then the number of such arrangements is

(1) less than 500
(2) at least 500 but less than 750
(3) at least 750 but less than 1000
(4) at least 1000

Sol: **(4)**

4 novels can be selected from 6 novels in ${}^6C_4$ ways. 1 dictionary can be selected from 3 dictionaries in ${}^3C_1$ ways. As the dictionary selected is fixed in the middle, the remaining 4 novels can be arranged in 4! ways.

$\therefore$ The required number of ways of arrangement = ${}^6C_4 \times {}^3C_1 \times 4! = 1080$

70. $\int_0^{\pi}[\cot x]\,dx$, $[\bullet]$ denotes the greatest integer function, is equal to

(1) $\frac{\pi}{2}$
(2) 1
(3) -1
(4) $-\frac{\pi}{2}$

Sol: **(4)**

Let $I=\int_0^{\pi}[\cot x]\,dx$ ...(1)

$=\int_0^{\pi}[\cot(\pi-x)]\,dx=\int_0^{\pi}[-\cot x]\,dx$ ...(2)

Adding (1) and (2)

$2I=\int_0^{\pi}[\cot x]\,dx+\int_0^{\pi}[-\cot x]\,dx==\int_0^{\pi}(-1)\,dx$ $\left[\begin{array}{l}\because [x]+[-x]=-1 \text{ if } x\notin Z \\ \quad\quad = 0 \text{ if } x\in Z\end{array}\right]$

$=[-x]_0^{\pi}=-\pi$

$\therefore I=-\frac{\pi}{2}$

71. For real x, let $f(x) = x^3 + 5x + 1$, then

(1) f is one-one but not onto R
(2) f is onto R but not one-one
(3) f is one-one and onto R
(4) f is neither one-one nor onto R

Sol: (3)

Given $f(x) = x^3 + 5x + 1$

Now $f'(x) = 3x^2 + 5 > 0,\ \forall x \in R$

$\therefore$ f(x) is strictly increasing function

$\therefore$ It is one-one

Clearly, f(x) is a continuous function and also increasing on R,

$\underset{x \to -\infty}{\text{Lt}} f(x) = -\infty$ and $\underset{x \to \infty}{\text{Lt}} f(x) = \infty$

$\therefore$ f(x) takes every value between $-\infty$ and ∞.

Thus, f(x) is onto function.

72. In a binomial distribution $B\left(n, p=\frac{1}{4}\right)$, if the probability of at least one success is greater than or equal to $\frac{9}{10}$, then n is greater than

(1) $\frac{1}{\log_{10}^{4}-\log_{10}^{3}}$ (2) $\frac{1}{\log_{10}^{4}+\log_{10}^{3}}$

(3) $\frac{9}{\log_{10}^{4}-\log_{10}^{3}}$ (4) $\frac{4}{\log_{10}^{4}-\log_{10}^{3}}$

Sol: **(1)**

$1-q^{n} \geq \frac{9}{10} \Rightarrow \left(\frac{3}{4}\right)^{n} \leq \frac{1}{10} \Rightarrow n \geq -\log_{\frac{3}{4}} 10 \Rightarrow n \geq \frac{1}{\log_{10}^{4}-\log_{10}^{3}}$

*73. If P and Q are the points of intersection of the circles $x^2+y^2+3x+7y+2p-5=0$ and $x^2+y^2+2x+2y-p^2=0$, then there is a circle passing through P, Q and (1, 1) for

(1) all values of p (2) all except one value of p

(3) all except two values of p (4) exactly one value of p

Sol: **(1)**

Given circles $S = x^2+y^2+3x+7y+2p-5=0$

$S' = x^2+y^2+2x+2y-p^2=0$

Equation of required circle is $S+\lambda S'=0$

As it passes through (1, 1) the value of $\lambda = -(7+2p)/(6-p^2)$

If 7 + 2p = 0, it becomes the second circle

$\therefore$ it is true for all values of p

74. The projections of a vector on the three coordinate axis are 6, - 3, 2 respectively. The direction cosines of the vector are

(1) $6, -3, 2$ (2) $\frac{6}{5}, -\frac{3}{5}, \frac{2}{5}$

(3) $\frac{6}{7}, -\frac{3}{7}, \frac{2}{7}$ (4) $-\frac{6}{7}, -\frac{3}{7}, \frac{2}{7}$

Sol: **(3)**

Projection of a vector on coordinate axis are $x_2-x_1, y_2-y_1, z_2-z_1$

$x_2-x_1=6, y_2-y_1=-3, z_2-z_1=2$

$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2} = \sqrt{36+9+4} = 7$

The D.C's of the vector are $\frac{6}{7}, -\frac{3}{7}, \frac{2}{7}$

*75. If $\left|Z-\frac{4}{z}\right|=2$, then the maximum value of $|Z|$ is equal to

(1) $\sqrt{3}+1$
(2) $\sqrt{5}+1$
(3) 2
(4) $2+\sqrt{2}$

Sol: **(2)**

$$|Z|=\left|\left(Z-\frac{4}{Z}\right)+\frac{4}{Z}\right| \Rightarrow |Z|=\left|Z-\frac{4}{Z}+\frac{4}{Z}\right|$$

$$\Rightarrow |Z| \le \left|Z-\frac{4}{Z}\right|+\frac{4}{|Z|} \Rightarrow |Z| \le 2+\frac{4}{|Z|}$$

$$\Rightarrow |Z|^2-2|Z|-4 \le 0$$

$$\left(|Z|-\left(\sqrt{5}+1\right)\right)\left(|Z|-\left(1-\sqrt{5}\right)\right) \le 0 \Rightarrow 1-\sqrt{5} \le |Z| \le \sqrt{5}+1$$

*76. Three distinct points A, B and C are given in the 2 – dimensional coordinate plane such that the ratio of the distance of any one of them from the point (1, 0) to the distance from the point (- 1, 0) is equal to $\frac{1}{3}$. Then the circumcentre of the triangle ABC is at the point

(1) (0, 0)
(2) $\left(\frac{5}{4}, 0\right)$
(3) $\left(\frac{5}{2}, 0\right)$
(4) $\left(\frac{5}{3}, 0\right)$

Sol: **(3)**

$P=(1, 0)$; $Q(-1, 0)$

Let $A=(x, y)$

$$\frac{AP}{AQ}=\frac{BP}{BQ}=\frac{CP}{CQ}=\frac{1}{3} \quad ..(1)$$

$$\Rightarrow 3AP=AQ \Rightarrow 9AP^2=AQ^2 \Rightarrow 9(x-1)^2+9y^2=(x+1)^2+y^2$$

$$\Rightarrow 9x^2-18x+9+9y^2=x^2+2x+1+y^2 \Rightarrow 8x^2-20x+8y^2+8=0$$

$$\Rightarrow x^2+y^2-5x+1=0 \quad ...(2)$$

$\therefore$ A lies on the circle

Similarly B, C are also lies on the same circle

$\therefore$ Circumcentre of ABC = Centre of Cricle (1) = $\left(\frac{5}{2}, 0\right)$

*77. The remainder left out when $8^{2n}-(62)^{2n+1}$ is divided by 9 is

(1) 0
(2) 2
(3) 7
(4) 8

Sol: **(2)**

$$8^{2n}-(62)^{2n+1}=(1+63)^n-(63-1)^{2n+1}$$

$$=(1+63)^n+(1-63)^{2n+1}=\left(1+{}^nc_1 63+{}^nc_2(63)^2+....+(63)^n\right)+\left(1-{}^{(2n+1)}c_1 63+{}^{(2n+1)}c_2(63)^2+....+(-1)(63)^{(2n+1)}\right)$$

$$=2+63\left({}^nc_1+{}^nc_2(63)+....+(63)^{n-1}-{}^{(2n+1)}c_1+{}^{(2n+1)}c_2(63)+....-(63)^{(2n)}\right)$$

$\therefore$ Reminder is 2

*78. The ellipse $x^2+4y^2=4$ is inscribed in a rectangle aligned with the coordinate axes, which in turn in inscribed in another ellipse that passes through the point (4, 0). Then the equation of the ellipse is

(1) $x^2+16y^2=16$ (2) $x^2+12y^2=16$

(3) $4x^2+48y^2=48$ (4) $4x^2+64y^2=48$

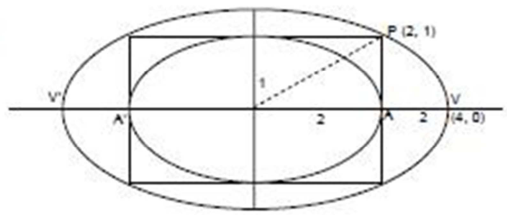
Sol: **(2)**

$x^2+4y^2=4 \Rightarrow \frac{x^2}{4}+\frac{y^2}{1}=1 \Rightarrow a=2, b=1 \Rightarrow P=(2,1)$

Required Ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{x^2}{4^2}+\frac{y^2}{b^2}=1$

(2, 1) lies on it

$\Rightarrow \frac{4}{16}+\frac{1}{b^2}=1 \Rightarrow \frac{1}{b^2}=1-\frac{1}{4}=\frac{3}{4} \Rightarrow b^2=\frac{4}{3}$



$\therefore \frac{x^2}{16}+\frac{y^2}{\left(\frac{4}{3}\right)}=1 \Rightarrow \frac{x^2}{16}+\frac{3y^2}{4}=1 \Rightarrow x^2+12y^2=16$

*79. The sum to the infinity of the series $1+\frac{2}{3}+\frac{6}{3^2}+\frac{10}{3^3}+\frac{14}{3^4}+......$ is

(1) 2
(2) 3
(3) 4
(4) 6

Sol: **(2)**

Let $S=1+\frac{2}{3}+\frac{6}{3^2}+\frac{10}{3^3}+\frac{14}{3^4}+....$ ...(1)

$\frac{1}{3}S=\frac{1}{3}+\frac{2}{3^2}+\frac{6}{3^3}+\frac{10}{3^4}+....$ ...(2)

Dividing (1) & (2)

$S\left(1-\frac{1}{3}\right)=1+\frac{1}{3}+\frac{4}{3^2}+\frac{4}{3^3}+\frac{4}{3^4}+....$

$\frac{2}{3}S=\frac{4}{3}+\frac{4}{3^2}\left(1+\frac{1}{3}+\frac{1}{3^2}+......\right) \Rightarrow \frac{2}{3}S=\frac{4}{3}+\frac{4}{3^2}\left(\frac{1}{1-\frac{1}{3}}\right)=\frac{4}{3}+\frac{4}{3^2}\frac{3}{2}=\frac{4}{3}+\frac{2}{3}=\frac{6}{2} \Rightarrow \frac{2}{3}S=\frac{6}{3} \Rightarrow S=3$

80. The differential equation which represents the family of curves $y=c_1e^{c_2x}$, where c_1 and c_2 are arbitrary constants is

(1) $y'=y^2$
(2) $y''=y'y$
(3) $yy''=y'$
(4) $yy''=(y')^2$

Sol: **(4)**

$y=c_1e^{c_2x}$ ...(1)

$y'=c_2c_1e^{c_2x}$

$y'=c_2y$ ...(2)

$y''=c_2y'$

From (2)

$c_2=\frac{y'}{y}$

So, $y''=\frac{(y')^2}{y} \Rightarrow yy''=(y')^2$

81. One ticket is selected at random from 50 tickets numbered 00, 01, 02, ..., 49. Then the probability that the sum of the digits on the selected ticket is 8, given that the product of these digits is zero, equals

(1) $\frac{1}{14}$
(2) $\frac{1}{7}$
(3) $\frac{5}{14}$
(4) $\frac{1}{50}$

Sol: **(1)**

S = { 00, 01, 02, ..., 49 }

Let A be the even that sum of the digits on the selected ticket is 8 then

A = { 08, 17, 26, 35, 44 }

Let B be the event that the product of the digits is zero

B = { 00, 01, 02, 03, ..., 09, 10, 20, 30, 40 }

$A \cap B=\{8\}$

Required probability = $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{50}}{\frac{14}{50}} = \frac{1}{14}$

82. Let y be an implicit function of x defined by $x^{2x} - 2x^x \cot y - 1 = 0$. Then $y'(1)$ equals

(1) – 1
(2) 1
(3) log 2
(4) – log 2

Sol: **(1)**

$x^{2x} - 2x^x \cot y - 1 = 0$ ...(1)

Now x = 1,

$1 - 2\cot y - 1 = 0 \Rightarrow \cot y = 0 \Rightarrow y = \frac{\pi}{2}$

Now differentiating eq. (1) w.r.t. 'x'

$2x^{2x}(1+\log x) - 2\left[x^x(-\operatorname{cosec}^2 y)\frac{dy}{dx} + \cot y\, x^x(1+\log x)\right] = 0$

Now at $\left(1, \frac{\pi}{2}\right)$

$2(1+\log 1) - 2\left(1(-1)\left(\frac{dy}{dx}\right)_{\left(1,\frac{\pi}{2}\right)} + 0\right) = 0$

$\Rightarrow 2 + 2\left(\frac{dy}{dx}\right)_{\left(1,\frac{\pi}{2}\right)} = 0 \Rightarrow \left(\frac{dy}{dx}\right)_{\left(1,\frac{\pi}{2}\right)} = -1$

83. The area of the region bounded by the parabola $(y-2)^2 = x-1$, the tangent to the parabola at the point (2, 3) and the x-axis is

(1) 3
(2) 6
(3) 9
(4) 12

Sol: **(3)**

Equation of tangent at (2, 3) to $(y-2)^2 = x-1$ is $S_1 = 0$

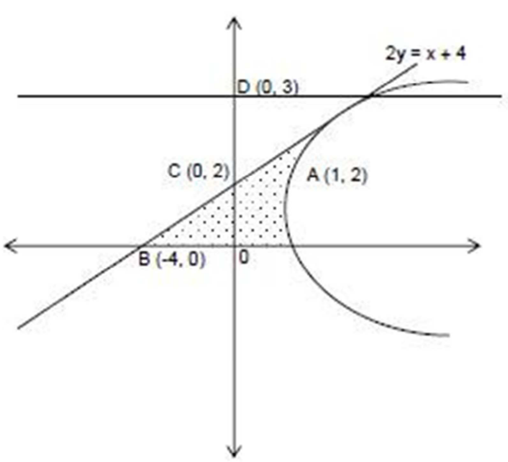
$\Rightarrow x - 2y + 4 = 0$

Required Area = Area of ΔOCB + Area of OAPD – Area of ΔPCD

$= \frac{1}{2}(4 \times 2) + \int_0^3 (y^2 - 4y + 5)\, dy - \frac{1}{2}(1 \times 2)$

$= 4 + \left[\frac{y^3}{3} - 2y^2 + 5y\right]_0^3 - 1 = 4 - 9 - 18 + 15 - 1$

$= 28 - 19 = 9$ sq. units



(or)

Area = $\int_0^3 (2y - 4 - y^2 + 4y - 5)\, dy = \int_0^3 (-y^2 + 6y - 5)\, dy = -\int_0^3 (3-y)^2\, dy = \left[\frac{(y-3)^3}{3}\right]_0^3 = \frac{27}{3} = 9$ sq.units

84. Given $P(x) = x^4 + ax^3 + bx^2 + cx + d$ such that x = 0 is the only real root of $P'(x) = 0$. If $P(-1) < P(1)$, then in the interval $[-1, 1]$

(1) $P(-1)$ is the minimum and $P(1)$ is the maximum of P

(2) $P(-1)$ is not minimum but $P(1)$ is the maximum of P

(3) $P(-1)$ is the minimum and $P(1)$ is not the maximum of P

(4) neither $P(-1)$ is the minimum nor $P(1)$ is the maximum of P

Sol: **(2)**

$P(x) = x^4 + ax^3 + bx^2 + cx + d$

$P'(x) = 4x^3 + 3ax^2 + 2bx + c$

$\because$ x = 0 is a solution for $P'(x) = 0$, $\Rightarrow c = 0$

$\therefore P(x) = x^4 + ax^3 + bx^2 + d$ ...(1)

Also, we have $P(-1) < P(1)$

$\Rightarrow 1 - a + b + d < 1 + a + b + d \Rightarrow a > 0$

$\because P'(x) = 0$, only when x = 0 and P(x) is differentiable in (- 1, 1), we should have the maximum and minimum at the points x = - 1, 0 and 1 only

Also, we have $P(-1) < P(1)$

$\therefore$ Max. of P(x) = Max. { P(0), P(1) } & Min. of P(x) = Min. { P(-1), P(0) }

In the interval [0 , 1],

$P'(x) = 4x^3 + 3ax^2 + 2bx = x(4x^2 + 3ax + 2b)$

$\because P'(x)$ has only one root x = 0, $4x^2 + 3ax + 2b = 0$ has no real roots.

$\therefore (3a)^2 - 32b < 0 \Rightarrow \frac{3a^2}{32} < b$

$\therefore b > 0$

Thus, we have a > 0 and b > 0

$\therefore P'(x) = 4x^3 + 3ax^2 + 2bx > 0, \forall x \in (0, 1)$

Hence P(x) is increasing in [0, 1]

$\therefore$ Max. of P(x) = P(1)

Similarly, P(x) is decreasing in [-1 , 0]

Therefore Min. P(x) does not occur at x = - 1

85. The shortest distance between the line $y - x = 1$ and the curve $x = y^2$ is

(1) $\frac{3\sqrt{2}}{8}$ (2) $\frac{2\sqrt{3}}{8}$

(3) $\frac{3\sqrt{2}}{5}$ (4) $\frac{\sqrt{3}}{4}$

Sol: **(1)**

$x - y + 1 = 0$ ...(1)

$x = y^2$

$1 = 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2y}$ = Slope of given line (1)

$\frac{1}{2y} = 1 \Rightarrow y = \frac{1}{2} \Rightarrow y = \frac{1}{2} \Rightarrow x = \left(\frac{1}{2}\right)^2 = \frac{1}{4} \Rightarrow (x, y) = \left(\frac{1}{4}, \frac{1}{2}\right)$

$\therefore$ The shortest distance is $\frac{\left|\frac{1}{4} - \frac{1}{2} + 1\right|}{\sqrt{1+1}} = \frac{3}{4\sqrt{2}} = \frac{3\sqrt{2}}{8}$

84. Given $P(x) = x^4 + ax^3 + bx^2 + cx + d$ such that x = 0 is the only real root of $P'(x) = 0$. If $P(-1) < P(1)$, then in the interval $[-1, 1]$

(1) $P(-1)$ is the minimum and $P(1)$ is the maximum of P
(2) $P(-1)$ is not minimum but $P(1)$ is the maximum of P
(3) $P(-1)$ is the minimum and $P(1)$ is not the maximum of P
(4) neither $P(-1)$ is the minimum nor $P(1)$ is the maximum of P

Sol: **(2)**

$P(x) = x^4 + ax^3 + bx^2 + cx + d$

$P'(x) = 4x^3 + 3ax^2 + 2bx + c$

$\because$ x = 0 is a solution for $P'(x) = 0$, $\Rightarrow c = 0$

$\therefore P(x) = x^4 + ax^3 + bx^2 + d$ ...(1)

Also, we have $P(-1) < P(1)$

$\Rightarrow 1 - a + b + d < 1 + a + b + d \Rightarrow a > 0$

$\because P'(x) = 0$, only when x = 0 and P(x) is differentiable in (- 1, 1), we should have the maximum and minimum at the points x = - 1, 0 and 1 only

Also, we have $P(-1) < P(1)$

$\therefore$ Max. of P(x) = Max. { P(0), P(1) } & Min. of P(x) = Min. { P(-1), P(0) }

In the interval [0 , 1],

$P'(x) = 4x^3 + 3ax^2 + 2bx = x(4x^2 + 3ax + 2b)$

$\because P'(x)$ has only one root x = 0, $4x^2 + 3ax + 2b = 0$ has no real roots.

$\therefore (3a)^2 - 32b < 0 \Rightarrow \frac{3a^2}{32} < b$

$\therefore b > 0$

Thus, we have a > 0 and b > 0

$\therefore P'(x) = 4x^3 + 3ax^2 + 2bx > 0, \forall x \in (0, 1)$

Hence P(x) is increasing in [0, 1]

$\therefore$ Max. of P(x) = P(1)

Similarly, P(x) is decreasing in [-1 , 0]

Therefore Min. P(x) does not occur at x = - 1

85. The shortest distance between the line $y - x = 1$ and the curve $x = y^2$ is

(1) $\frac{3\sqrt{2}}{8}$
(2) $\frac{2\sqrt{3}}{8}$
(3) $\frac{3\sqrt{2}}{5}$
(4) $\frac{\sqrt{3}}{4}$

Sol: **(1)**

$x - y + 1 = 0$ ...(1)

$x = y^2$

$1 = 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2y}$ = Slope of given line (1)

$\frac{1}{2y} = 1 \Rightarrow y = \frac{1}{2} \Rightarrow y = \frac{1}{2} \Rightarrow x = \left(\frac{1}{2}\right)^2 = \frac{1}{4} \Rightarrow (x, y) = \left(\frac{1}{4}, \frac{1}{2}\right)$

$\therefore$ The shortest distance is $\frac{\left|\frac{1}{4} - \frac{1}{2} + 1\right|}{\sqrt{1+1}} = \frac{3}{4\sqrt{2}} = \frac{3\sqrt{2}}{8}$

Directions: Question number 86 to 90 are Assertion – Reason type questions. Each of these questions contains two statements

Statement-1 (Assertion) and Statement-2 (Reason).
Each of these questions also have four alternative choices, only one of which is the correct answer. You have to select the correct choice

86. Let $f(x) = (x+1)^2 - 1, x \geq -1$

Statement-1 : The set $\{x : f(x) = f^{-1}(x)\} = \{0, -1\}$

Statement-2 : f is a bijection.

(1) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(2) Statement-1 is true, Statement-2 is true; Statement-2 is **not** a correct explanation for Statement-1
(3) Statement-1 is true, Statement-2 is false
(4) Statement-1 is false, Statement-2 is true

Sol: **(3)**
There is no information about co-domain therefore f(x) is not necessarily onto.

87. Let $f(x) = x|x|$ and $g(x) = \sin x$.

Statement-1 : gof is differentiable at x = 0 and its derivative is continuous at that point.
Statement-2 : gof is twice differentiable at x = 0.

(1) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(2) Statement-1 is true, Statement-2 is true; Statement-2 is **not** a correct explanation for Statement-1
(3) Statement-1 is true, Statement-2 is false
(4) Statement-1 is false, Statement-2 is true

Sol: **(3)**

$f(x) = x|x|$ and $g(x) = \sin x$

$$gof(x) = \sin(x|x|) = \begin{cases} -\sin x^2 & ,x < 0 \\ \sin x^2 & ,x \geq 0 \end{cases}$$

$$\therefore (gof)'(x) = \begin{cases} -2x\cos x^2 & ,x < 0 \\ 2x\cos x^2 & ,x \geq 0 \end{cases}$$

Clearly, $L(gof)'(0) = 0 = R(gof)'(0)$

$\therefore$ gof is differentiable at x = 0 and also its derivative is continuous at x = 0

$$\text{Now, } (gof)''(x) = \begin{cases} -2\cos x^2 + 4x^2\sin x^2 & ,x < 0 \\ 2\cos x^2 - 4x^2\sin x^2 & ,x \geq 0 \end{cases}$$

$\therefore L(gof)''(0) = -2$ and $R(gof)''(0) = 2$

$\therefore L(gof)''(0) \neq R(gof)''(0)$

$\therefore$ gof(x) is not twice differentiable at x = 0.

*88. Statement-1 : The variance of first n even natural numbers is $\frac{n^2-1}{4}$

Statement-2 : The sum of first n natural numbers is $\frac{n(n+1)}{2}$ and the sum of squares of first n natural numbers is $\frac{n(n+1)(2n+1)}{6}$

(1) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(2) Statement-1 is true, Statement-2 is true; Statement-2 is **not** a correct explanation for Statement-1
(3) Statement-1 is true, Statement-2 is false
(4) Statement-1 is false, Statement-2 is true

Sol: **(4)**
Statement-2 is true

Statement-1:
Sum of n even natural numbers = n (n + 1)

$\text{Mean}(\bar{x}) = \frac{n(n+1)}{n} = n+1$

$\text{Variance} = \left[\frac{1}{n}\sum(x_i)^2\right] - (\bar{x})^2 = \frac{1}{n}\left[2^2 + 4^2 + \ldots\ldots + (2n)^2\right] - (n+1)^2$

$= \frac{1}{n}2^2\left[1^2 + 2^2 + \ldots\ldots + n^2\right] - (n+1)^2 = \frac{4}{n}\frac{n(n+1)(2n+1)}{6} - (n+1)^2$

$= \frac{(n+1)\left[2(2n+1) - 3(n+1)\right]}{3} = \frac{(n+1)\left[4n+2-3n-3\right]}{3} = \frac{(n+1)(n-1)}{3} = \frac{n^2-1}{3}$

$\therefore$ Statement 1 is false.

89. Statement-1 : $\sim(p \leftrightarrow \sim q)$ is equivalent to $p \leftrightarrow q$.

Statement-2 : $\sim(p \leftrightarrow \sim q)$ is a tautology.

(1) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(2) Statement-1 is true, Statement-2 is true; Statement-2 is **not** a correct explanation for Statement-1
(3) Statement-1 is true, Statement-2 is false
(4) Statement-1 is false, Statement-2 is true

Sol: **(3)**

p	q	$p \leftrightarrow q$	~q	$p \leftrightarrow \sim q$	$\sim(p \leftrightarrow \sim q)$
T	T	T	F	F	T
T	F	F	T	T	F
F	T	F	F	T	F
F	F	T	T	F	T

90. Let A be a 2 x 2 matrix

Statement-1 : $\text{adj}(\text{adj } A) = A$

Statement-2 : $|\text{adj } A| = |A|$

(1) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(2) Statement-1 is true, Statement-2 is true; Statement-2 is **not** a correct explanation for Statement-1
(3) Statement-1 is true, Statement-2 is false
(4) Statement-1 is false, Statement-2 is true

Sol: **(2)**

$|\text{adj } A| = |A|^{n-1} = |A|^{2-1} = |A|$

$\text{adj}(\text{adj } A) = |A|^{n-2} A = |A|^0 A = A$