IIT-JEE-Mathematics–1997

Time : Three Hours **Max. Marks : 100**

Instructions :

1. Answer all questions in the language of your choice as shown in your admit card.
2. The paper consists of seven printed pages (17 questions).
3. Answer to next question should start after drawing a separating horizontal line with a space of 3cm.
4. All sub-questions of a question should be answered at one place in the same order as in the question paper.
5. There is no negative marking.
6. Use of all type of calculating devices, Graph paper, Logarithmic/Trigonometric/ Statistical Tables is prohibited.

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1. There are five sub-questions in this question. For answering each sub-question, four alternatives are given, and only one of them is correct. Indicate your answer for each sub-question by writing one of the letters A, B, C or D ONLY in the answer-book.

(i) If $g(x)=\int_0^x \cos^4 t\, dt$, then $g(x+p)$ equals :

(A) g (x)+ g (π) (B) g (x)- g (π)
(C) g (x)g (π) (D) (g (x))/(g (x))

(ii) If f (x)= x / sinx and g (x)= x / tanx , where 0 < x £ 1, then in this interval :
(A) Both f (x) and g (x) are increasing functions.
(B) Both f (x) and g (x) are decreasing functions.
(C) f (x) is an increasing function.
(D) g (x) is an increasing function.

(iii) The parameter, on which the value of the determinant does not depend upon is :

$$\begin{vmatrix} 1 & a & a^2 \\ \cos(p-d)x & \cos px & \cos(p+d) \\ \sin(p-d)x & \sin px & \sin(p+d) \end{vmatrix}$$

(A) a (B) p
(C) d (D) x

(iv) The graph of the function cos x (x + 2) − cos2 (x + 1) is:

(A) a straight line passing through (0, − sin2 1) with slope 2
(B) a straight line passing through (0, 0)
(C) a parabola with vertex (1, − sin2, 1)
(D) a straight line passing through the point (π/2,-sin2 1) and parallel to the x-axis.

(v)

$$\lim_{n\to\infty} \frac{1}{n} \sum_{1}^{2n} \frac{r}{\sqrt{n^2 + r^2}} \; equals:$$

(A) 1 + √5 (B) − 1+ √5
(C) − 1 + √2 (D) 1 + √2

2. A tangent to the ellipse x2 + 4y2 = 4 meets the ellipse x2 + 2y2 = 6 at P and Q. Prove that the tangents at P and Q of the ellipse x2 + 2y2 = 6 are at right angles

3. Prove that the values of the function $\frac{\sin x \ \cos 3x}{\sin 3x \ \cos \ x}$ do not lie between $\frac{1}{3}$ and 3 for any real x.

4. Determine the value of $\int_{-\pi}^{r} \frac{2x\,(1+\sin x)}{1+cos^2\, x}\, dx$

5. Let a + b = 4, where a < 2, and let g (x) be a differentiable function. If $\frac{dg}{dx} > 0$ for all x, prove that $\int_0^a g(x)\, dx + \int_0^b g(x)\, dx$ increases as (b – a) increases.

6. The question contained FIVE incomplete statements. Complete these statements so that the resulting statements are correct. Write ONLY the answers in your answer-book in the order in which the statement are given below.
(i) The sum of all the real roots of the equations |x-2|2+|x-2|-2=0 is ….. ……… .

(ii) Let p and q be roots of the equations x2 − 2x + A = 0 and let r and s be the roots of the equation x2 − 18 + B = 0. If p < q < r < s are in arithmetic progression the A = …………..
and B = …………

(iii) Let $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = 10\,\vec{a} + 2\vec{b}$, and $\overrightarrow{OC} = \vec{b}$, where O, A and C are non-collinear points. Let p denote the area of the quadrilateral OABC, and let q denote the area of the parallelogram with OA and OC as adjacent sides. If p = kq, then k= ……………

(iv) The sum of the rational terms in the expansion of $\left(\sqrt{2} + 3^{1/5}\right)^{10}$ is …………

(v) For each natural number k, let C_k denote the circle with radius K centimeters an centre at the origin. On the circle C_k, a particle moves K centimeters in the counter-clockwise direction. After completing its motion on Ck the particle moves to C_{k+1} in the radial direction. The motion of the particle continues in this manner. The particle starts at (1, 0). If the particle crosses the positive direction of the a-axis for the first time on the circle C_n then n = ………………

7. Let $0 < A_i < \pi$ for i = 1, 2, ………, n. Use mathematical induction to prove that $\sin A_1 + \sin A_2 + \ldots\ldots + \sin A_n \le n \sin\left(\frac{A_1 + A_2 + \ldots\ldots + A_n}{n}\right)$ where $n \ge 1$ is a natural number.
{You may use the fact that $p \sin x + (1 - p) \sin y \le \sin [px + (1 - p\, y]$, where $0 \le p \le 1$ and $0 \le x, y \le \pi$.}

8. Determine the values of x for which the following function fails to be continuous or differentiable:

$$f(x) = \begin{cases} 1 - x, & x < 1 \\ (1 - x)(2 - x) & 1 \le x \le 2 \\ 3 - x & x > 2 \end{cases}$$

Justify your answer.

9. If $\vec{A}, \vec{B}$ and $\vec{C}$ are vectors such that $|\vec{B}| = |\vec{C}|$. Prove that

$$[(\vec{A} + \vec{B}) \times (\vec{A} + \vec{C})] \times (\vec{B} \times \vec{C}).(\vec{B} + \vec{C}) = 0$$

10. Let z1 and z2 be roots of the equation z2 + pz + q = 0, where the co-efficients p and q may be complex numbers. Let A and B represent z1 and z2 in the complex plane.
If If ÐAOB = α ¹ 0 and OA = OB, where O is the origin, prove that p2 = 4q cos2 (α/2).

11. The question contains FIVE incomplete statements. Complete these statements so that the resulting statements are correct. Write ONLY the answers in your answer-book in the order in which the statements are given below.

(i) Let $f(x)$ be a continuous function defined for $1 \le x \le 3$. If $f(x)$ takes rational values for all x and $f(2) = 10$, then $f(1.5)$ = ……………….

(ii) The value of $\int_1^{e^{37}} \frac{\pi \sin(\pi \ln x)}{x} dx$ is ……………

(iii) Let $\frac{d}{dx} F(x) = \frac{e^{\sin x}}{x}, x > 0$. If $\int_1^4 \frac{2e^{\sin x^2}}{x} dx = F(k) - F(1)$ then one of the possible values of k is …………………… .

(iv) The real roots of the equation $\cos^7 x + \sin^4 x = 1$ in the interval $(-\pi, \pi)$ are ………., ………… and ………. .

(vi) The chords of contact of the pair of tangents drawn from each point on the line 2x + = 4 to the circle $x^2 + y^2 = 1$ pass through the point ………

12. If p and q are chosen randomly from the set {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, with replacement, determine the probability that the roots of the equation x2 + px + q = 0 are real.

13. Let u (x) and v (x) satisfy the differential equations du/dx + p (x) u = f (x) and dv/dx + p (x) v = g (x), where p (x) and g (x) are continuous functions. If u (x1) > v (x1) for some x1 and f (x) > g (x) for all x > x1, prove that any point (x, y) where x > x1 does not satisfy the equations y = u (x) and y = v (x).

14. Let S be a square of unit area. Consider any quadrilateral which has one vertex on each side of S. If a, b, c and d denote the lengths of the sides of the quadrilateral, prove that 2 £ a2 + b2 + c2 + d2 £ 4.

15. Let f (x) = Maximum {x2, (1 − x)2, 2x (1 − x)}, where 0 £ x £ 1. Determine the area of the region bounded by the curves y = f (x), x-axis, x = 0 and x = 1.

16. Prove that ∑k-1n-1 (n-k) cos(n/2), where n ³ 3 is an integer.

17. Let C be any circle with centre (0,√2). Prove that at the most two rational points can be there on C.

(A rational point is a point both of whose coordinates are rational numbers.)