IIT-JEE-Mathematics–1999

Time : 3 hours Max. Marks : 200

GENERAL INSTRUCTIONS

1. Do not break the seal of the question paper before you are instructed to do so by the invigilator.
2. This question paper is in two sections. Section I has 35 objective type questions. Section II has 12 regular questions.
3. Answer Section I only the special machine-gradable OBJECTIVE RESONSE SHEET (ORS) that is inserted in this booklet. Questions of Section I will not be graded if answered anywhere else.
4. Answer problems of Section II in the answer-book.
5. Without breaking the seal of this booklet, take out the Response Sheet (ORS) for Section I. Make sure that the ORS has the SAME QUESTION PAPER CODE, printed on it as on the top of this page.
6. Write your name, registration number and name of the Centre at the specified locations on the right half of the ORS for Section I.
7. Using a soft HB pencil darken the appropriate bubble under each digit of your registration number.
8. The Objective Response Sheet will be collected back after 75 minutes have expired from the start of the examination. In case you finish Section I before the expiry of 75 minutes, you may start answering Section II.

INSTRUCTIONS FOR SECTION SECTION I

1. Questions are to answered by darkening with a soft HB pencil the appropriate bubble (marked A, B, C or D) against the question number on the left-hand side of the Objective Response Sheet.
2. In case you wish to change an answer, erase the old answer completely using a good soft erases.
3. The answer sheet will be collected back after 75 minutes from the start of the examination.
4. There is no negative marking.
5. Question numbers 1-25 carry 2 marks each and have only one correct answer.
6. Question Numbers 26-35 carry 3 marks each, and may have more than one correct answers. All the correct answers must be marked in these questions to get any credit.

IMPORTANT INFORMATION

1. Use of logarithmic tables is not allowed.
2. Use of calculator is not allowed.

SECTION I

DIRECTIONS: Select the most appropriate alternative A, B, C or D in questions 1-25.

1. If I = $\sqrt{-1}$, then $4 + 5\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{365}$ is equal to:

(A) $1 - \sqrt{3}$ (B) $-1 + i\sqrt{3}$

(C) $i\sqrt{3}$ (D) $-i\sqrt{3}$

3. If x1, x2, x3 as well as y1, y2, y3 are in G.P. with the same common ratio, then the points (x1, y1), (x2, y2) and (x3, y3):
(A) lie on a straight line
(B) lie on the ellipse
(C) lie on a circle
(D) are vertices of a triangle

3. If the function *f* : [1, ¥) --> [1, -¥) is defined by *f* (x) = 2x(x−1), then *f*−1 (x) is :
(A) (1/2)x(x-1)

(B) 1/2 (1+√(1+4 log_2 x))
(C) 1/2(1-√(1+4 log2 x))
(D) not defined

4. The harmonic mean of the roots of the equation :
(5 + √2 x2 – (4 + √5) x + 8 + 2√5 = 0 is
(A) 2
(B) 4
(C) 6
(D) 8

5. The function *f* (x) = sin4 x + cos4 x increases if :
(A) 0 < x < π/8
(B) π/4 < x < 3π/8
(C) 3π/8 < x < 5π/8
(D) 5π/8 < x < 3π/4

6. The curve described parametrically by x = t2 + t + 1, y = t2− t + 1 represents :
(A) a pair of straight lines
(B) an ellipse
(C) a parabola
(D) a hyperbola

7. In a triangle PQR, ÐR = π/2. If tan (P/2) and tan (Q/2) are the roots of the equation ax2 + bx + c = 0 (a ¹ 0), then :
(A) a + b = c

(B) b + c = a
(C) a + c = b
(D) b = c

8. If for a real number y, [y] is the greatest integer less than or equal to y, then the value of the integral ∫π/2 3π/2 [2 sinx. dx] is:
(A) -π
(B) 0
(C) -π/2
(D) π/2

9. Let a1, a2, ……, a10 be in A.P. and h1, h2, ……, h10 be in H.P. If a1 = h1 = 2 and a10 = h10 = 3, then a4 h7 is :
(A) 2
(B) 3
(C) 5
(D) 6

10. Let $\vec{a} = 2\hat{\imath} + \hat{\jmath} - 2\hat{k}$ and $\hat{b} = \hat{\imath} + \hat{\jmath}$. If $\hat{c}$ is a vector such that $\vec{a}.\vec{c} = |\vec{c} - \vec{a}| = 2\sqrt{2}$ and the angle between $(\vec{a} \times \vec{b})$ and $\vec{c}$ is 30^0, then $|(\vec{a} \times \vec{b}) \times \vec{c}|=$

(A) $\frac{3}{2}$ (B) $\frac{3}{2}$

(C) 2 (D) 3

11. The number of real solutions of
tan–1 √(x(x+1)) + sin–1 √(x2+x+1)=π/2 is:
(A) zero
(B) one
(C) two
(D) infinite

12. Let P (a sec q, b tan q) and Q (a sec q, b tan q) where q + q = π/2, be two points on the hyperbola x2/a2 -y2/b2 = 1. If (h, k) is the point of intersection of the normals at P and Q, then K is equal to :
(A) (a2+b2)/a
(B) -((a2+b2)/a)
(C) (a2+b2)/b
(D) -((a2+b2)/b)

13. Let PQR be a right angled isosceles triangle, right angled at P (2, 1). If the equation of the line QR is 2x + y = 3, then the equation representing the pair of lines PQ and PR is :
(A) 3x2 − 3y2 + 8xy + 20x + 10y + 25 = 0
(B) 3x2 − 3y2 + 8xy − 20x − 10y + 25 = 0

(C) 3x2 – 3y2 + 8xy + 10x + 15y + 20 = 0
(D) 3x2 – 3y2 – 8xy – 10x – 15y – 20 = 0

14. If f(x) $\begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1) \end{vmatrix}$ then f (100) is equal to :

(A) 0 (B) 1
(C) 100 (D) –100

15. The function f(x) = [x]2 – [x2] (where [y] is the greatest integer less than or equal to y), is discontinuous at :
(A) all integers
(B) all integers except 0 and 1
(C) all integers except 0
(D) all integers except 1

16. If two distinct chords, drawn from the point (p, q) on the circle x2+ y2=px + qy (where pq ¹ 0) are bisected by the x-axis, then :
(A) p2 = q2
(B) p2 = 8q2
(C) p2 <8q2
(D) p2 >8q2

17. The function f(x) = (x2 – 1)|x2 – 3x + 2|is NOT differentiable at:

(A) –1
(B) 0
(C) 1
(D) 2

18. If the roots of the equation x2 – 2ax + a2 + a – 3 = 0 are real and less than 3,then:
(A) a <2
(B) 2 < a < 3
(C) 3 <a < 4
(D) a >4

19. A solution of the differential equation (dx/dy)2-x (dx/dy) + y = 0 is :

(A) y = 2
(B) y = 2x
(B) y = 2x – 4
(D) y = 2x2 – 4

20. limx→0 (x tan2x - 2x tanx)/(1-cos2x)2 is :

(A) y = 2
(B) y = 2x
(C) y = 2x – 4
(D) y = 2x2 – 4

21. Let $\vec{a} = \vec{2}\hat{j} + \hat{k}, \vec{b} = \hat{\imath} + \vec{2}\hat{j} - \hat{k}$ and a unit vector $\vec{c}$ be coplanar. If $\vec{c}$ is perpendicular to $\vec{a}$, then $\vec{c}$ =

(A) $\frac{1}{\sqrt{2}}(-\hat{j} + \hat{k})$
(B) $\frac{1}{\sqrt{3}}(-\hat{\imath} - \hat{j} - \hat{k})$
(C) $\frac{1}{\sqrt{5}}(\hat{\imath} - 2\hat{j})$
(D) $\frac{1}{\sqrt{3}}(\hat{\imath} - \hat{j} - \hat{k})$

22. If in the expansion of (1 + x)n, the coefficients of x and x2 are 3 and –6 respectively, then m is :
(A) 6
(B) 9
(C) 12
(D) 24

23. ∫π/33π/4 dx/(1+cosx) is equal to :

(A) 2
(B) –2
(C) 1/2
(D) -1/2

24. If x = 9 is the chord of contact of the hyperbola x2 – y2 = 0, then the equation of the corresponding pair of tangents is :
(A) 9x2 – 8y2 + 18x – 9 = 0
(B) 9x2 – 8y2 – 18x + 9 = 0
(C) 9x2 – 8y2 – 18x – 9 = 0
(D) 9x2 – 8y2 + 18x + 9 = 0

25. If the integers m and n are chosen at random between 1 and 100, then the probability that a number of the form 7m + 7n is divisible by 5 equals :
(A) 1/4
(B) 1/7
(C) 1/8
(D) 1/49

DIRECTIONS : Question numbers 26-35 carry 3 marks each and may have more than one correct answers. All correct answers must be marked to get any credit in these questions.

26. Let L1 be a straight line passing through the origin and L2 be the straight line x + y = 1. If the intercepts made by the circle x2 + y2 − x + 3y = 0 on L1 and L2 are equal, then which of the following equations can represent L1?
(A) x + y = 0
(B) x − y = 0
(B) x + 7y = 0
(D) x − 7y = 0

27. Let $\vec{a}$ and $\vec{b}$ be two non-collinear unit vectors. If $\vec{u} = \vec{a} - (\vec{a}.\vec{b})\vec{b}$ and $\vec{v} = \vec{a} \times \vec{b}$, then $|\vec{v}|$ is:
(A) $|\vec{u}|$
(B) $|\vec{u}| + |\vec{u}.\vec{a}|$
(C) $|\vec{u}| + |\vec{u}.\vec{b}|$
(D) $|\vec{u}| + \vec{u}.(\vec{a}+\vec{b})$

28. For a positive integer n, let a (n) = 1 + 1/2 + 1/3 + 1/4 +.......+ 1/((2n)-1). Then :
(A) a (100) £ 100
(B) a (100) > (100)
(C) a (200) £ 100
(D) a (200) > 100

29. The function f(x) = ∫-1xt (et-1)(t-1)(t-2)3 (t-3)5dt has a local minimum at x =

(A) 0
(B) 1
(C) 2
(D) 3

30. On the ellipse 4x2 + 9y2 = 1, the points at which the tangents are parallel to the line 8x = 9y are :
(A) (2/5,1/5)
(B) (-2/5,1/5)
(C) (-2/5,-1/5)
(D) (2/5,-1/5)

31. The probabilities that a student passes in Mathematics, Physics and Chemistry are m, p and c, respectively. Of these subjects, the student has a 75% chance of passing in atleast one, a 50% chance of passing in atleast two, and a 40% chance of passing in exactly two. Which of the following relations are true?
(A) p + m + c = 19/20
(B) p + m + c = 27/20
(C) pmc = 1/10
(D) pmc = 1/4

32. The differential equation representing the family of curves y2 = 2c (x + √c), where c is a positive parameter, is of :

(A) order 1
(B) order 2
(C) degree 3
(D) 10

33. Let S1, S2 … be squares such that for each n ³ 1, the length of a side of Snequals the length of a diagonal of Sn+1. If the length of a side of S1 is 10 cm, then for which of the following values of n is the area of Sn less than 1 sq. cm?
(A) 7
(B) 8
(C) 9
(D) 10

34. For which of the following values of m is the area of the region bounded by the curve y = x − x2 and he line y = mx equals 9/2?
(A) −4
(B) −2
(C) 2
(D) 4

35. For a positive integer n., let f_n (θ)= (tan θ/2) (1 + secθ) (1 + sec2θ) (1 + sec 4θ) … (1 + sec 2nθ). Then

(A) f2 (π/16)=1 ;
(B) f3 (π/32)=1
(C) f4 (π/16)=1
(D) f5 (π/128)=1

SECTION II

Instructions
There 12 questions in the section. Attempt ALL questions.
At the end of the anwers to a question, draw a horizontal line and start answer to the next question. The corresponding question number must be written in the left margin. Answer all parts of a question at one place only.
The use of Arabic numerals (0, 1, 2,…….9) only is allowed in answering the questions irrespective of the language in which you answer.

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1. For complex numbers z and q, prove that |z|2 w − |w|2 z = z − w if and only if z = w or z w- = 1.

2. Let a, b, c d be real numbers in G.P. If u, v, w satisfy the system of equations
u + 2v + 3 w = 6
4u + 5v + 6 w = 12
6u + 9v = 4
Then slow that the roots of the equation :

(1/u+1/v+1/w)x2 + [(b – c)2 + (c – a)2 + (d – b)2] x + u + v + w = 0
and 20x2 + 10 (a – d)2 x – 9 = 0 are reciprocals of each other.

3. Let n be any positive integer. Prove that :

$$\sum_{k=0}^{m} \frac{\binom{2n-k}{k}}{\binom{2n-k}{n}} \cdot \frac{(2n-4k+1)}{(2n-2k+1)} 2^{n-2k} = \frac{\binom{n}{m}}{\binom{2n-2m}{n-m}} 2^{n-2m},$$

for each nonnegative integer m ≤ n. $\left(Here \binom{p}{q} = {}^{p}C_{q}\right)$

4. Let ABC be a triangle having O and I as its circumcentre and incentre respectively. If R and r are the circumradius and the inradius, respectively, then prove that (IO)2 = R2 – 2Rr. Further show that the triangle BIO is a right-angled triangle if and only if b is the arithmetic mean of a and c.

5. Let T1, T2 be two tangents drawn from (–2, 0) onto the circle C : x2 + y2 = 1. Determine the circles touching C and having T1, T2 as their pair of tangents. Further, find the equations of all possible common tangents to these circles, when taken two at a time.

6. Consider the family of circles x2 + y2 = r2, 2 < r < 5. If in the first quadrant, the common tangent to a circle of this family and the ellipse 4x2 + 25y2 = 100 meets the co-ordinate axes at A and B, then find the equation of the locus of the mid point of AB.

7. Integrate

(A) $\int \frac{x^3 + 3x + 2}{(x^2+1)^2 (x+1)} dx.$

(B) $\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}} dx.$

8. Let f(x) be a continuous function given by

$$f(x) = \begin{cases} 2x, & |x| \le 1 \\ x^2 + ax + b, & |x| > 1 \end{cases}$$

Find the area of the region in the third quadrant bounded by the curves x = –2y2and y = f(x) lying on the left on the line 8x + 1 = 0.

9. Find the co-ordinates of all the P on the ellipse x2/a2 +y2/b2 = 1, for which the area of the triangle PON is maximum, where O denotes the origin and N, the foot of the perpendicular from O to the tangent at P.

10. A curve passing through the point (1, 1) has the property that the perpendicular distance of the origin from the normal at any point P of the curve is equal to the distance of P from the x-axis. Determine the equation of the curve.

11. Eight players P1, P2, …… P8 play a knock-out tournament. It is known that whenever the players Pi and Pj play, the play Pi will win if i < j. Assuming that the players are paired at random in each round, what is the probability that the player P4 reaches the final?

12. Let $\vec{u}$ and $\vec{v}$ be unit vectors. If $\vec{w}$ is a vector such that $\vec{w} + (\vec{w} \times \vec{u}) = \vec{v}$, then prove that $|(\vec{u} \times \vec{v}).\vec{w}| \leq \frac{1}{2}$ and that the equality holds if and only if $\vec{u}$ is perpendicular to $\vec{v}$.