

Matrices and Determinants - Solutions

Sol 1.

We should have,

$$\begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 3 & -4 \end{vmatrix} = 0$$

$$\Rightarrow 1(-4k + 6) - k(-12 + 4) + 3(9 - 2k) = 0$$

$$\Rightarrow -2k + 33 = 0$$

$$\Rightarrow k = 33/2$$

Substituting $k = 33/2$ and putting $x = b$, where $b \in \mathbb{Q}$, we get the system as

$$33y + 6z = -2b \dots\dots\dots (1)$$

$$33y - 4z = -6b \dots\dots\dots (2)$$

$$3y - 4z = -2b \dots\dots\dots (3)$$

$$(1) - (2) \Rightarrow 10z = 4b \Rightarrow z = 2/5 b$$

$$(1) \Rightarrow 33y = -2b - 12b/5 = -22b/5 \Rightarrow y = -2b/15$$

\therefore The solution is $x = b, y = -2b/15, z = 2b/5$

Sol 2.

The given determinant, on expanding along R_1 , we get

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = a(bc - a^2) - b(b^2 - ac) + c(ab - c^2)$$

$$= 3abc - a^3 - b^3 - c^3 = -(a^3 + b^3 + c^3 - 3abc)$$

$$= -(a + b + c) [a^2 + b^2 + c^2 - ab - bc - ca]$$

$$= -1/2 (a + b + c) [2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca]$$

$$= -1/2 (a + b + c) [(a - b)^2 + (b - c)^2 + (c - a)^2]$$

As $a, b, c > 0$

$$\therefore a + b + c > 0$$

Also $a \neq b \neq c \therefore (a - b)^2 + (b - c)^2 + (c - a)^2 > 0$ Hence the given determinant is - ve.

Sol 3.

$$\begin{vmatrix} x^2 + x & x+1 & x-2 \\ 2x^2 + 3x - 1 & 3x & 3x-3 \\ x^2 + 2x + 3 & 2x-1 & 2x-1 \end{vmatrix} = xA + B$$

$$\text{L. H. S.} = \begin{vmatrix} x^2 + x & x+1 & x-2 \\ 2x^2 + 3x - 1 & 3x & 3x-3 \\ x^2 + 2x + 3 & 2x-1 & 2x-1 \end{vmatrix}$$

Operation $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - R_1$

$$\begin{vmatrix} x^2 + x & x+1 & x-2 \\ x-1 & x-2 & x+1 \\ x+3 & x-2 & x+1 \end{vmatrix}$$

$$= \begin{vmatrix} x^2 & x+1 & x-2 \\ 0 & x-2 & x+1 \\ 0 & x-2 & x+1 \end{vmatrix} + \begin{vmatrix} x & x+1 & x-2 \\ x-1 & x-2 & x+1 \\ x+3 & x-2 & x+1 \end{vmatrix}$$

$$= 0 + \begin{vmatrix} x & x+1 & x-2 \\ x-1 & x-2 & x+1 \\ x+3 & x-2 & x+1 \end{vmatrix}$$

Operating $R_3 \rightarrow R_3 - R_2$ and $R_2 \rightarrow R_2 - R_1$

$$= \begin{vmatrix} x & x+1 & x-2 \\ -1 & -3 & 3 \\ 4 & 0 & 0 \end{vmatrix} = \begin{vmatrix} x & x & x \\ -1 & -3 & 3 \\ 4 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & -2 \\ -1 & -3 & 3 \\ 4 & 0 & 0 \end{vmatrix}$$

$$= x \begin{vmatrix} 1 & 1 & 1 \\ -1 & -3 & 3 \\ 4 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & -2 \\ -1 & -3 & 3 \\ 4 & 0 & 0 \end{vmatrix}$$

= $xA + B = \text{R. H. S.}$ Hence Proved.

Sol 4.

The given system of equations is

$$3x - y + 4z = 3$$

$$x + 2y - 3z = -2$$

$$6x + 5y + \lambda z = -3$$

$$\text{Then } D = \begin{vmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{vmatrix}$$

$$= 3(\lambda + 15) + 1(\lambda + 18) + 4(5 - 12) = 7\lambda + 35 = 7(\lambda + 5)$$

For unique sol., $D \neq 0$

$$\therefore \lambda + 5 \neq 0 \text{ or } \lambda \neq -5$$

\therefore System has unique Sol For $\lambda \neq -5$

For $\lambda = -5$, $D = 0$

Then

$$D_1 = \begin{vmatrix} 3 & -1 & 4 \\ -2 & 2 & -3 \\ -3 & 5 & -5 \end{vmatrix}$$

$$= 3(-10 + 15) + 1(10 - 9) + 4(-10 + 6)$$

$$= 15 + 1 - 16 = 0$$

$$D_2 = \begin{vmatrix} 3 & 3 & 4 \\ 1 & -2 & -3 \\ 6 & -3 & -5 \end{vmatrix}$$

$$= 3(10 - 9) - 3(-5 + 18) + 4(-3 + 12) = 3 - 39 + 36 = 0$$

$$D_3 = \begin{vmatrix} 3 & -1 & 3 \\ 1 & 2 & -2 \\ 6 & -5 & -3 \end{vmatrix}$$

$$= 3(-6 + 10) + 1(-3 + 12) + 3(5 - 12) = 12 + 9 - 21 = 0$$

$$D_1 = D_2 = D_3 = 0$$

\therefore Infinite many solutions.

Let $z = k$ then equations becomes

$$3x - y = 3 - 4k$$

$$x + 2y = 3k - 2$$

on solving we get

$$x = 4 - 5k/7, y = 13k - 9/7, z = k$$

This satisfies the third equation.

Sol 5.

On L. H. S. = D, applying operations $C_2 \rightarrow C_2 + C_1$ and $C_3 \rightarrow C_3 + C_2$ and using ${}^n C_r + {}^n C_{r+1} = {}^{n+1} C_{r+1}$, we get

$$D = \begin{vmatrix} {}^x C_r & {}^{x+1} C_{r+1} & {}^{x+1} C_{r+2} \\ {}^y C_r & {}^{y+1} C_{r+1} & {}^{y+1} C_{r+2} \\ {}^= C_r & {}^{=+1} C_{r+1} & {}^{=+1} C_{r+2} \end{vmatrix}$$

Operating $C_3 + C_2$ and using the same result, we get

$$D = \begin{vmatrix} {}^x C_r & {}^{x+1} C_{r+1} & {}^{x+2} C_{r+2} \\ {}^y C_r & {}^{y+1} C_{r+1} & {}^{y+2} C_{r+2} \\ {}^= C_r & {}^{=+1} C_{r+1} & {}^{=+2} C_{r+2} \end{vmatrix} = \text{RHS}$$

Hence Proved

Sol 6.

The system will have a non – trivial solution if

$$\begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} = 0$$

Expanding along C_1 , we get

$$\Rightarrow (28 - 21) \sin 3\theta - (-7 - 7) \cos 2\theta + 2(-3 - 4) = 0$$

$$\Rightarrow 7 \sin 3\theta + 14 \cos 2\theta - 14 = 0$$

$$\Rightarrow \sin 3\theta + 2 \cos 2\theta - 2 = 0$$

$$\Rightarrow 3 \sin \theta - 4 \sin^3 \theta + 2(1 - 2 \sin^2 \theta) - 2 = 0$$

$$\Rightarrow 4 \sin^3 \theta + 4 \sin^2 \theta - 3 \sin \theta = 0$$

$$\Rightarrow \sin \theta (2 \sin \theta - 1) (2 \sin \theta + 3) = 0$$

$$\sin \theta = 0 \text{ or } \sin \theta = 1/2$$

$$(\sin \theta = -3/2 \text{ not possible})$$

$$\Rightarrow \theta = n\pi \text{ or } \theta = n\pi + (-1)^n \pi/6, n \in \mathbb{Z}.$$

Sol 7.

We have

$$\Delta a = \begin{vmatrix} a-1 & n & 6 \\ (a-1)^2 & 2n^2 & 4n-2 \\ (a-1)^3 & 3n^2 & 3n^2-3n \end{vmatrix}$$

NOTE THIS STEP

$$\text{Then } \sum_{a=1}^n \Delta a = \begin{vmatrix} (1-1) & n & 6 \\ (1-1)^2 & 2n^2 & 4n-2 \\ (1-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix} + \begin{vmatrix} (2-1) & n & 6 \\ (2-1)^2 & 2n^2 & 4n-2 \\ (2-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix} + \dots$$

$$+ \begin{vmatrix} (n-1) & n & 6 \\ (n-1)^2 & 2n^2 & 4n-2 \\ (n-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$$

$$= \begin{vmatrix} 1+2+3+\dots+(n-1) & n & 6 \\ 1^2+2^2+3^2+\dots+(n-1)^2 & 2n^2 & 4n-2 \\ 1^3+2^3+3^3+\dots+(n-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$$

$$= \begin{vmatrix} \frac{n(n-1)}{2} & n & 6 \\ \frac{n(n-1)(2n-1)}{6} & 2n^2 & 4n-2 \\ \left(\frac{n(n-1)}{2}\right)^2 & 3n^3 & 3n^2-3n \end{vmatrix}$$

$$= \frac{n^2(n-1)}{12} \begin{vmatrix} 6 & 1 & 6 \\ 2(2n-1) & 2n & 2(2n-1) \\ 3n(n-1) & 3n^2 & 3n(n-1) \end{vmatrix}$$

(Taking $n(n-1)/12$ common from C_1 and n from C_2)

= 0 (as C_1 and C_3 are identical)

$$\text{Thus, } \sum_{a=1}^n \Delta a = 0 \Rightarrow \sum_{a=1}^n \Delta a = c \text{ (a constant)}$$

Where $c = 0$

Sol 8.

Given that A, B, C are integers between 0 and 9 and the three digit numbers A 28, 3 B9 and 62C are divisible by a fixed integer k.

$$\text{Now, } D = \begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$$

On operating $R_2 \rightarrow R_2 + 10 R_3 + 100 R_1$, we get

$$= \begin{vmatrix} A & 3 & 6 \\ A28 & 3B9 & 62C \\ 2 & B & 2 \end{vmatrix} = \begin{vmatrix} A & 3 & 6 \\ kn_1 & kn_2 & kn_3 \\ 2 & B & 2 \end{vmatrix}$$

[As per question A28, 3 B9 and 62 C are divisible by k, therefore,

$$A 28 = kn_1$$

$$3 B9 = kn_2$$

$$62 C = kn_3]$$

$$= k \begin{vmatrix} A & 3 & 6 \\ n_1 & n_2 & n_3 \\ 2 & B & 2 \end{vmatrix} = k \times \text{some integral value.}$$

$\Rightarrow D$ is divisible by k.

Sol 9.

$$\text{Consider } \begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$$

Operating $R_1 \rightarrow R_1 - R_2$ and $R_2 \rightarrow R_2 - R_3$ we get

$$\begin{vmatrix} p-a & -(q-b) & c \\ 0 & q-b & c-r \\ a & b & r \end{vmatrix} = 0$$

Taking $(p - q)$, $(q - b)$ and $(r - c)$ common from $(C_1, C_2$ and C_3 resp. we get

$$\Rightarrow (p-a)(q-b)(r-c) \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ \frac{a}{p-a} & \frac{b}{q-b} & \frac{r}{r-c} \end{vmatrix} = 0$$

Expanding along R_1

$$\Rightarrow (p-a)(q-b)(r-c) [1(r/r-c + b/q-b) + a/p-a] = 0$$

As given that $p \neq a, q \neq b, r \neq c$

$$\therefore r/r-c + b/q-b + a/p-a = 0$$

$$\Rightarrow r/r-c + q - (q-b)/q-b + p - (p-a)/p-a = 0$$

$$\Rightarrow r/r-c + q/q-b - 1 + p/p-a - 1 = 0$$

$$\Rightarrow p/p-a + q/q-b + r/r-c = 2$$

Sol 10.

$$D = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$$

$$= n!(n+1)!(n+2)! \begin{vmatrix} 1 & n+1 & (n+2)(n+1) \\ 1 & n+2 & (n+3)(n+2) \\ 1 & n+3 & (n+4)(n+3) \end{vmatrix}$$

Operating $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_2$, we get

$$D = (n!)^3 (n+1)^2 (n+2) \begin{vmatrix} 1 & n+1 & n^2 + 3n + 2 \\ 0 & 1 & 2n+4 \\ 0 & 1 & 2n+6 \end{vmatrix}$$

Operating $R_3 \rightarrow R_3 - R_2$

$$D = (n!)^3 (n+1)^2 (n+2) \begin{vmatrix} 1 & n+1 & n^2 + 3n + 2 \\ 0 & 1 & 2n+4 \\ 0 & 0 & 2 \end{vmatrix}$$

$$= (n!)^3 (n+1)^2 (n+2) \cdot 1 \cdot [2]$$

$$\Rightarrow D/(n!)^3 = 2(n+1)^2(n+2)$$

$$\Rightarrow D/(n!)^3 - 4 = 2(n^3 + 4n^2 + 5n + 2) - 4$$

$$= 2(n^3 + 4n^2 + 5n) = 2n(n^2 + 4n + 5)$$

$\Rightarrow D/(n!)^3 - 4$ is divisible by n .

Sol 11.

Given that $\lambda, \alpha \in \mathbb{R}$ and system of linear equations

$$\lambda x + (\sin \alpha) y + (\cos \alpha) z = 0$$

$$x + (\cos \alpha) y + (\sin \alpha) z = 0$$

$$-x (\sin \alpha) y - (\cos \alpha) z = 0$$

Has a non trivial solution, therefore $D = 0$

$$\Rightarrow \begin{vmatrix} \lambda & \sin \alpha & \cos \alpha \\ 1 & \cos \alpha & \sin \alpha \\ -1 & \sin \alpha & -\cos \alpha \end{vmatrix} = 0$$

$$\Rightarrow \lambda (-\cos^2 \alpha - \sin^2 \alpha) - \sin \alpha (-\cos \alpha + \sin \alpha) + \cos \alpha (\sin \alpha + \cos \alpha) = 0$$

$$\Rightarrow -\lambda + \sin \alpha \cos \alpha - \sin^2 \alpha + \sin \alpha \cos \alpha + \cos^2 \alpha = 0$$

$$\Rightarrow \lambda = \cos^2 \alpha - \sin^2 \alpha + \sin \alpha \cos \alpha$$

$$\Rightarrow \lambda = \cos 2\alpha + \sin 2\alpha$$

$$\text{For } \lambda = 1, \cos 2\alpha + \sin 2\alpha = 1$$

$$1/\sqrt{2} \cos 2\alpha + 1/\sqrt{2} \sin 2\alpha = 1/\sqrt{2}$$

$$\Rightarrow \cos 2\alpha \cos \pi/4 + \sin 2\alpha \sin \pi/4 = 1/\sqrt{2}$$

$$\Rightarrow \cos(2\alpha - \pi/4) = \cos \pi/4$$

$$\Rightarrow 2\alpha - \pi/4 = 2n\pi \pm \pi/4$$

$$\Rightarrow 2\alpha = 2n\pi + \pi/4 + \pi/4; 2n\pi - \pi/4 + \pi/4$$

$$\Rightarrow \alpha = n\pi + \pi/4 \text{ or } n\pi$$

Sol 12.

$$\text{L. H. S} = \begin{vmatrix} \cos A \cos P + \sin A \sin P & \cos(A-Q) & \cos(A-R) \\ \cos B \cos P + \sin B \sin P & \cos(B-Q) & \cos(B-R) \\ \cos C \cos P + \sin C \sin P & \cos(C-Q) & \cos(C-R) \end{vmatrix}$$

$$= \cos P = \begin{vmatrix} \cos A & \cos(A-Q) & \cos(A-R) \\ \cos B & \cos(B-Q) & \cos(B-R) \\ \cos C & \cos(C-Q) & \cos(C-R) \end{vmatrix} + \sin P \begin{vmatrix} \sin A & \cos(A-Q) & \cos(A-R) \\ \sin B & \cos(B-Q) & \cos(B-R) \\ \sin C & \cos(C-Q) & \cos(C-R) \end{vmatrix}$$

Operating; $C_2 \rightarrow C_2 - C_1 (\cos Q)$; $C_3 \rightarrow C_3 - C_1 (\cos R)$ on first determinant and $C_2 \rightarrow C_3 - (\sin Q) C_1$ and $C_3 \rightarrow C_3 - (\sin R) C_1$ on second determinant, we get

$$= \cos P \begin{vmatrix} \cos A & \sin A \sin Q & \sin A \sin R \\ \cos B & \sin B \sin Q & \sin B \sin R \\ \cos C & \sin C \sin Q & \sin C \sin R \end{vmatrix} = \cos P \begin{vmatrix} \sin A & \cos A \cos Q & \cos A \cos R \\ \sin B & \cos B \cos Q & \cos B \cos R \\ \sin C & \cos C \cos Q & \cos C \cos R \end{vmatrix}$$

$$= \cos P \sin Q \sin R \begin{vmatrix} \cos A & \sin A & \sin A \\ \cos B & \sin B & \sin B \\ \cos C & \sin C & \sin C \end{vmatrix}$$

$$+ \sin P \cos Q \cos R \begin{vmatrix} \sin A & \cos A & \cos A \\ \sin B & \cos B & \cos B \\ \sin C & \cos C & \cos C \end{vmatrix}$$

$$= 0 + 0 \text{ [Both determinants become zero as } C_2 \equiv C_3] = 0 = \text{R. H. S.}$$

Sol 13.

Let us denote the given determinant by Δ . Taking

$1/a (a + d) (a + 2d)$ as common from

R_1 , $1/(a + d) (a + 2d) (a + 3d)$ from R_2 and

$1/(a + 2d) (a + 3d) (a + 4d)$ from R_3 we get

$$\Delta = 1/a (a + d)^2 (a + 2d)^3 (a + 3d)^2 (a + 4d) \Delta_1$$

Where

$$\Delta_1 = \begin{vmatrix} (a+d)(a+2d) & a+2d & a \\ (a+2d)(a+3d) & a+3d & a+d \\ (a+3d)(a+4d) & a+4d & a+2d \end{vmatrix}$$

Applying $R_3 \rightarrow R_3 - R_2$ and $R_2 \rightarrow R_2 - R_1$, we get

$$\Delta_1 = \begin{vmatrix} (a+d)(a+2d) & a+2d & a \\ (a+2d)(2d) & d & d \\ (a+3d)(2d) & d & d \end{vmatrix}$$

Applying $R_3 \rightarrow R_3 - R_2$, we get

$$\Delta_1 = \begin{vmatrix} (a+d)(a+2d) & a+2d & a \\ (a+2d)(2d) & d & d \\ 2d^2 & 0 & 0 \end{vmatrix}$$

Expanding along R_3 , we get

$$\Delta_1 = (2d)^2 \begin{vmatrix} a+2d & a \\ d & d \end{vmatrix} = (2d)^2 (d) (a+2d-a) = 4d^4$$

Thus, $\Delta = 4d^4/a (a+d)^2 (a+2d)^3 (a+3d)^2 (a+4d)$.

Sol14.

Given that a, b, c are p th, q th and r th terms of an H. P.

$\Rightarrow 1/a, 1/b, 1/c$ are p th, q th, r th terms of an A. P.

$$\left. \begin{aligned} \frac{1}{a} &= A + (p-1)D \\ \therefore \frac{1}{b} &= A + (q-1)D \\ \frac{1}{c} &= A + (r-1)D \end{aligned} \right\} \dots\dots\dots (1)$$

Now given determinant is

$$\Delta = \begin{vmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = abc \begin{vmatrix} \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

Substituting the values of $1/a, 1/b, 1/c$ from (1), we get

$$= abc \begin{vmatrix} A + (p-1)D & A + (q-1)D & A + (r-1)D \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} \quad \text{Operating } R_1 \rightarrow R_1 - (A-D) R_3 - D R_2, \text{ we get}$$

$$\Delta = abc \begin{vmatrix} 0 & 0 & 0 \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Sol 15.

$$R_2 \rightarrow R_2 + R_3,$$

$$= 0 \begin{vmatrix} \sin \theta & \cos \theta & \sin 2\theta \\ 2 \sin \theta \cos \frac{2\pi}{3} & 2 \cos \theta \cos \frac{2\pi}{3} & 2 \sin 2\theta \cos \frac{4\pi}{3} \\ \sin \left(\theta - \frac{2\pi}{3} \right) & \cos \left(\theta - \frac{2\pi}{3} \right) & \sin \left(2\theta - \frac{4\pi}{3} \right) \end{vmatrix}$$

$$= \begin{vmatrix} \sin \theta & \cos \theta & \sin 2\theta \\ -\sin \theta & -\cos \theta & -\sin 2\theta \\ \sin \left(\theta - \frac{2\pi}{3} \right) & \cos \left(\theta - \frac{2\pi}{3} \right) & \sin \left(2\theta - \frac{4\pi}{3} \right) \end{vmatrix} = 0$$

Sol 16.

Given that $A^T A = I$

$$\text{Where } A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} \therefore A^T = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$$

$$\therefore A^T A = I$$

$$\Rightarrow \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a^2 + b^2 + c^2 & ab + bc + ca & ab + bc + ca \\ ab + bc + ca & a^2 + b^2 + c^2 & ab + bc + ca \\ ab + bc + ca & ab + bc + ca & a^2 + b^2 + c^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow a^2 + b^2 + c^2 = 1 \dots\dots\dots (1)$$

$$\text{And } ab + bc + ca = 0 \dots\dots\dots (2)$$

$$abc = 1 \text{ (given)} \dots\dots\dots (3)$$

$$\text{Now } a^3 + b^3 + c^3 - 3abc$$

$$= (a + b + c) (a^2 + b^2 + c^2 - ab - bc - ca)$$

$$\Rightarrow a^3 + b^3 + c^3 = (a + b + c) [1 - 0] + 3 \times 1 \text{ [Using (1), (2) and (3)]}$$

$$\Rightarrow a^3 + b^3 + c^3 = a + b + c + 3$$

$$\text{Now } (a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$$

$$\Rightarrow (a + b + c)^2 = 1$$

$$\Rightarrow (a + b + c) = 1$$

[$\because (a + b + c) \neq -1$ as a, b, c all are +ve numbers]

\therefore we get

$$a^3 + b^3 + c^3 = 4$$

ALTERNATE SOLUTION

Given that $A^T A = I$

$$\Rightarrow |A^T A| = |A^T| |A| = |A| |A| = 1 \quad [\because |I| = 1]$$

$$\Rightarrow |A|^2 = 1 \dots\dots\dots (1)$$

$$\text{From given matrix } A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$$

$$|A| = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = a^3 + b^3 + c^3 - 3abc \dots\dots\dots (2)$$

$$\therefore (a^3 + b^3 + c^3 - 3abc = 1 \text{ or } -1$$

But for a^3, b^3, c^3 using $AM \geq GM$

$$\text{We get } a^3 + b^3 + c^3 / 3 \geq \sqrt[3]{a^3 b^3 c^3}$$

\therefore We must have

$$a^3 + b^3 + c^3 - 3abc = 1$$

$$\Rightarrow a^3 + b^3 + c^3 = 1 + 3 \times 1 = 4 \text{ [Using } abc = 1]$$

Sol 17.

We are given that $MM^T = I$ where M is a square matrix of order 3 and $\det M = 1$.

$$\text{Consider } \det (M - I) = \det (M - MM^T) \text{ [Given } MM^T = I]$$

$$= \det [M (I - M^T)] \quad [\because |AB| = |A| |B|]$$

$$= -(\det M) (\det (M^T - I))$$

$$= -1 [\det (M^T - I)] \quad [\because \det (M) = 1]$$

$$= -\det(M - I)$$

$$[\because \det M^T - I = \det [(M - I)^T] = \det (M - I)]$$

$$\Rightarrow 2 \det (M - I) = 0 \Rightarrow \det (M - I) = 0 \text{ Hence Proved}$$

Sol18.

$$\text{Given that, } A = \begin{bmatrix} a & 1 & 0 \\ 1 & b & d \\ 1 & b & c \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, U = \begin{bmatrix} f \\ g \\ h \end{bmatrix}$$

And $AX = U$ has infinite many solution.

$$\Rightarrow |A| = 0 = |A_1| = |A_2| = |A_3|$$

Now, $|A| = 0$

$$\Rightarrow \begin{vmatrix} a & 1 & 0 \\ 1 & b & d \\ 1 & b & c \end{vmatrix} = a(bc - bd) - 1(c - d) = 0$$

$$\Rightarrow (ab - 1)(c - d) = 0$$

$$\Rightarrow ab = 1 \text{ or } c = d \dots\dots\dots(1)$$

$$\text{And } |A_1| = \begin{vmatrix} f & 1 & 0 \\ g & b & d \\ h & b & c \end{vmatrix} = 0$$

$$\Rightarrow f(bc - bd) - 1(gc - hd) = 0$$

$$\Rightarrow f b (c - d) = gc - hd \dots\dots\dots(2)$$

$$|A_2| = \begin{vmatrix} a & 1 & f \\ 1 & b & g \\ 1 & b & h \end{vmatrix} = 0$$

$$\Rightarrow a(gc - hd) - f(c - d) = 0 \Rightarrow a(gc - hd) = f(c - d)$$

$$|A_3| = \begin{vmatrix} a & 1 & f \\ 1 & b & g \\ 1 & b & h \end{vmatrix} = 0$$

$$\Rightarrow a(bh - bg) - 1(h - g) + f(b - d) = 0$$

$$\Rightarrow ab(h - g) - (h - g) = 0$$

$$\Rightarrow ab = 1 \text{ or } h = g \dots\dots\dots(3)$$

Taking $c = d \Rightarrow h = g$ and $ab \neq 1$ (from (1), (2) and (3))

Now taking $BX = V$

$$\text{Where } B = \begin{vmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{vmatrix}, V = \begin{vmatrix} a^2 \\ 0 \\ 0 \end{vmatrix}$$

$$\text{Then } |B| = \begin{vmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{vmatrix} = 0$$

[∵ In view of $c = d$ and $g = h$, C_1 and C_2 and C_3 are identical

⇒ $BX = V$ has no unique solution

$$\text{And } |B_1| = \begin{vmatrix} a^2 & 1 & 1 \\ 0 & d & c \\ 0 & g & h \end{vmatrix} = 0 \quad (\because c = d, g = h)$$

$$|B_2| = \begin{vmatrix} a & a^2 & 1 \\ 0 & 0 & c \\ f & 0 & h \end{vmatrix} a^2 cf = a^2 df \quad (\because c = d)$$

$$|B_3| = \begin{vmatrix} a & 1 & a^2 \\ 0 & d & 0 \\ f & g & 0 \end{vmatrix} a^2 df$$

⇒ If $adf \neq 0$ then $|B_2| = |B_3| \neq 0$

Hence no solution exist.