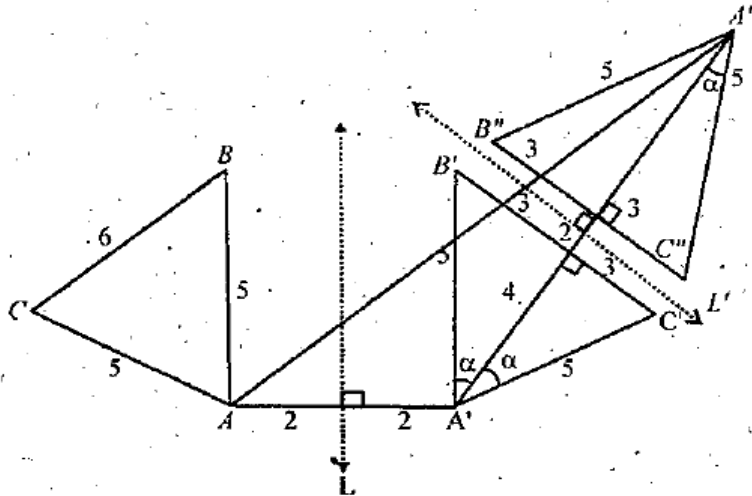


Properties of Triangle – Solutions

Sol 1.

Let L be the line parallel to side AB of ΔABC , at a distance of 2 cm from AB , in which the first reflection $\Delta A' B' C'$ is obtained. Let L' be the second line parallel to $B' C'$, at a distance of 2 cm from $B' C'$, in which reflection of $\Delta A' B' C'$ is taken as $\Delta A'' B'' C''$.

In figure, size of $\Delta A'' B'' C''$ is same to the size of $\Delta A' B' C'$.



From figure $AA' = 4\text{cm}$ and $A'A'' = 12\text{cm}$. So to find AA'' it suffices to know $\angle AA'A''$, clearly

$\angle AA'A'' = 90^\circ + \alpha$ where $\sin \alpha = 3/5$

$\Rightarrow \cos \angle AA'A'' = \cos (90^\circ + \alpha) = \sin \alpha = -3/5$ and hence

$$AA'' = \sqrt{(AA')^2 + (A'A'')^2 - 2AA' \times A'A'' \cdot \cos (90^\circ + \alpha)}$$

$$= \sqrt{16 + 144 + 96 \times 3/5}$$

$$= \sqrt{1088/5} = 8\sqrt{17/5}\text{ cm.}$$

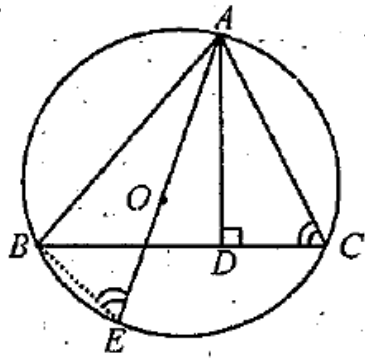
Sol 2.

(a) In radius of the circle is given by

$$r = (s - b) \tan B/2 = (a + b + c/2 - b) \tan \pi/4 = a + c - b/2$$

$$2r = a + c - b \Rightarrow \text{Diameter} = BC + AB - AC$$

(b) Given a ΔABC in which $AD \perp BC$, AE is diameter of circumcircle of ΔABC .



To prove:

$$AB \times AC = AE \times AD$$

Construction: Join BE

Proof: $\angle ABE = 90^\circ$ (\angle in a semi-circle)

Now in Δ 's ABE and ADC

$$\angle ABE = \angle ADC \quad (\text{each } 90^\circ)$$

$$\angle AEB = \angle ACD \quad (\angle\text{'s in the same segment})$$

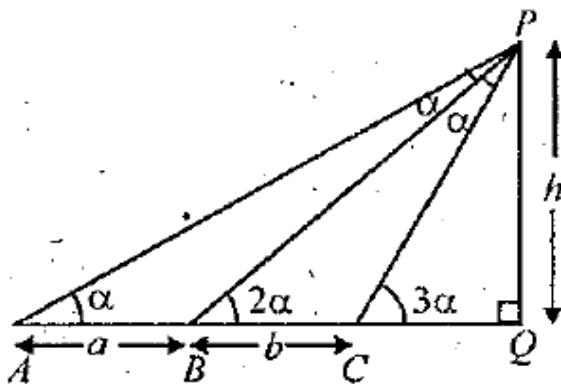
$$\therefore \Delta ABE \sim \Delta ADC \quad (\text{by AA similarity})$$

$$\Rightarrow AB/AD = AE/AC$$

$$\Rightarrow AB \times AC = AD \times AE \quad (\text{Proved})$$

Sol 3.

(a) By exterior angle theorem, in the adjacent fig.



$$\angle APB = \angle BPC = \alpha$$

Also in ΔABP , $\angle BAP = \angle APB = \alpha$

$$\Rightarrow AB = PB = a$$

Applying sine law in ΔPBC , we get

$$\Rightarrow a / \sin (180^\circ - 3\alpha) = b / \sin \alpha = PC / \sin 2\alpha \dots\dots\dots (1)$$

$$\Rightarrow a/3 \sin \alpha - 4 \sin^3 \alpha = b / \sin \alpha = PC/2 \sin \alpha \cos \alpha$$

$$\Rightarrow a/3 - 4 \sin^2 \alpha = b/1 = PC/2 \cos \alpha$$

$$\Rightarrow 3 - 4 \sin^2 \alpha = a/b \Rightarrow \sin^2 \alpha = 3b - a/4b$$

$$\Rightarrow \cos^2 \alpha = b + a/4b$$

$$\Rightarrow \cos \alpha = 1/2 \sqrt{b + a/b}$$

$$\text{Also } PC = 2b \cos \alpha = \sqrt{b} (a + b)$$

Now in ΔPCQ

$$\sin 3\alpha = h/PC \Rightarrow h = PC (a \sin \alpha/b) \text{ [Using eqn. (1)]}$$

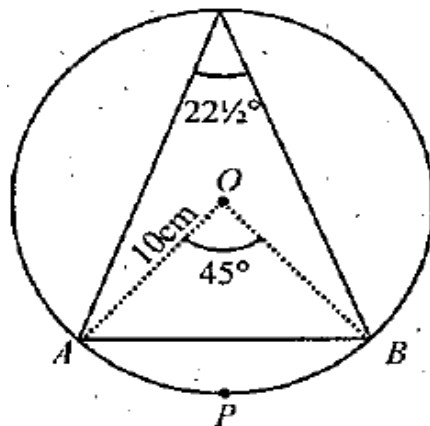
$$\Rightarrow h = \sqrt{b} (a + b) a/b \sqrt{\frac{3b - a}{4b}}$$

$$\Rightarrow h = a/2b \sqrt{(a + b) (3b - a)}$$

$$(b) \because \angle ACB = 22 \frac{1}{2}^\circ$$

$$\therefore \angle AOB = 45^\circ$$

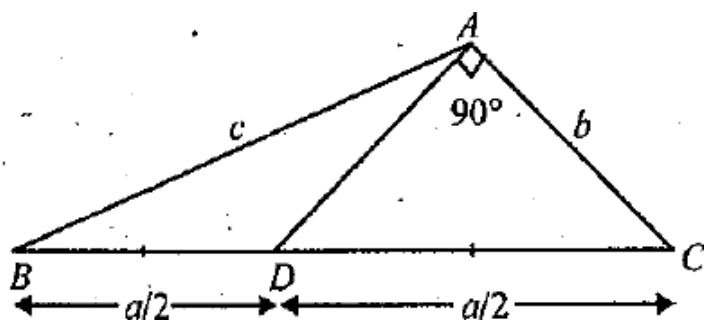
$$r = 10 \text{ cm}$$



Area of the segment APB = Area of the sector AOB - area of ΔAOB

$$= 1/8 \pi r^2 - 1/2 \times 10 \times 10 \sin 45^\circ \text{ (Using } \Delta = 1/2 bc \sin A) = 3.14 \times 100/8 - 50/\sqrt{2} = 3.91 \text{ sq. cm.}$$

Sol 4.



In ΔACD , $\cos C = b/a/2 = 2b/a$ (1)

In ΔABC , $\cos C = a^2 + b^2 + c^2/2ab$ (2)

From (1) and (2),

$2b/a = a^2 + b^2 - c^2/2ab \Rightarrow b^2 = 1/3 (a^2 - c^2)$ (3)

Also $\cos A = b^2 + c^2 - a^2/2bc$

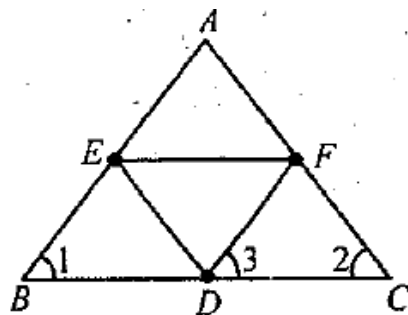
$\therefore \cos A \cos C = b^2 + c^2 - a^2/2bc \times 2b/a = b^2 + c^2 - a^2/ac$

$= \frac{\frac{1}{3}(a^2 - c^2) + (c^2 - a^2)}{ac} = 2(c^2 - a^2)/3ac$

Sol 5.

Given that $AB = AC$

$\therefore \angle 1 = \angle 2$ (1)



But $AB \parallel DF$ (given) and BC is transversal

$\therefore \angle 1 = \angle 3$ (2)

From equation (1) and (2)

$\angle 2 = \angle 3$

$\Rightarrow DF = CF$ (3)

Similarly we can prove

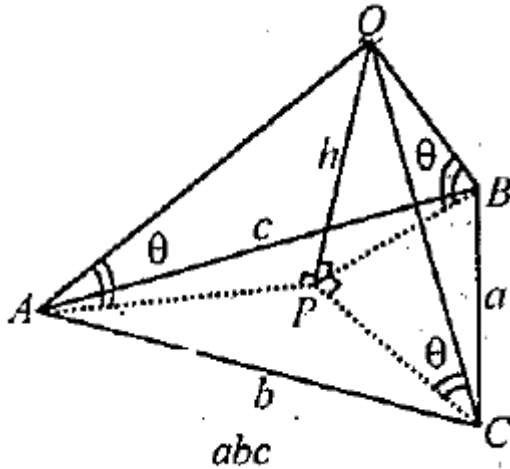
$$DE = BE$$

$$\begin{aligned} \text{Now, } DF + FA + AE + ED &= CF + FA + AE + BE \\ &= AC + AB \quad [\text{using equation (3) and (4)}] \end{aligned}$$

Sol 6.

(i) Let h be the height of tower PQ .

$$\text{In } \Delta APQ \tan \theta = h/AP \Rightarrow AP = h/\tan \theta$$



Similarly in Δ 's BPQ and CPQ we obtain

$$BP = h/\tan \theta = CP$$

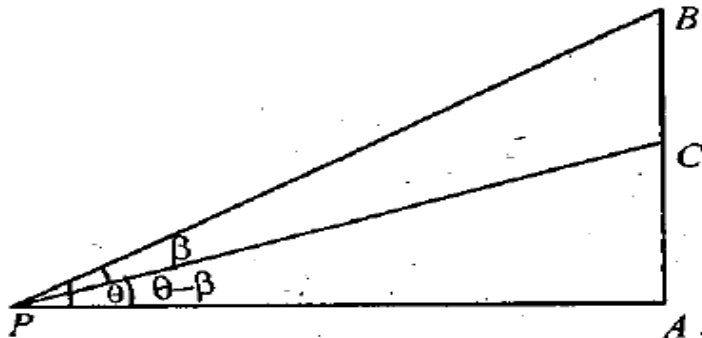
$$\therefore AP = BP = CP$$

$\Rightarrow P$ is the circum-centre of ΔABC with circum radius $R = AP = abc/4\Delta$

$$\therefore h = AP \tan \theta = abc \tan \theta / 4\Delta$$

(ii) Given $AP = AB \times n$

$$\Rightarrow AB/AP = 1/n = \tan \theta \therefore \tan \theta = 1/n$$



Also $\tan(\theta - \beta) = AC/AP = 1/2 AB/AP = 1/2n$

$\Rightarrow \frac{\tan \theta - \tan \beta}{1 + \tan \theta \tan \beta} = 1/2n \Rightarrow \frac{\frac{1}{n} \tan \beta}{1 + \frac{1}{n} \tan \beta} = 1/2n$

$\Rightarrow 2n - 2n^2 \tan \beta = n + \tan \beta$

$\Rightarrow (2n^2 + 1) \tan \beta = n \Rightarrow \tan \beta = n/2n^2 + 1$

Sol 7.

As the angles A, B, C of ΔABC is in AP

\therefore Let $A = x - d, B = x, C = x + d$

But $A + B + C = 180^\circ$ (\angle Sum prop. of Δ)

$\therefore x - d + x + x + d = 180^\circ$

$\Rightarrow 3x = 180^\circ \Rightarrow x = 60^\circ \therefore \angle B = 60^\circ$

Now by sine law in ΔABC , we have

$b/\sin B = c/\sin C \Rightarrow \sin B/\sin C$

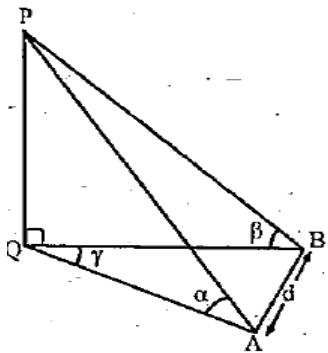
$\Rightarrow \sqrt{3}/\sqrt{2} = \sin 60^\circ/\sin C$ [using $b:c = \sqrt{3}:\sqrt{2}$ and $\angle B = 60^\circ$]

$\Rightarrow \sqrt{3}/\sqrt{2} = \sqrt{3}/2 \sin C \Rightarrow \sin C = 1/\sqrt{2} = \sin 45^\circ$

$\therefore \angle C = 45^\circ \Rightarrow \angle A = 180^\circ - (\angle B + \angle C) = 180^\circ - (60^\circ + 45^\circ) = 75^\circ$

Sol 8.

Let ht. of pole PQ be h.



In $\Delta APQ, \tan \alpha = h/AQ$

$\Rightarrow AQ = h/\tan \alpha$ (1)

In $\Delta BPQ, \tan \beta = h/BQ \Rightarrow BQ = h/\tan \beta$ (2)

In $\Delta ABQ, \cos \gamma = \frac{AQ^2 + BQ^2 - AB^2}{2 AQ BQ}$

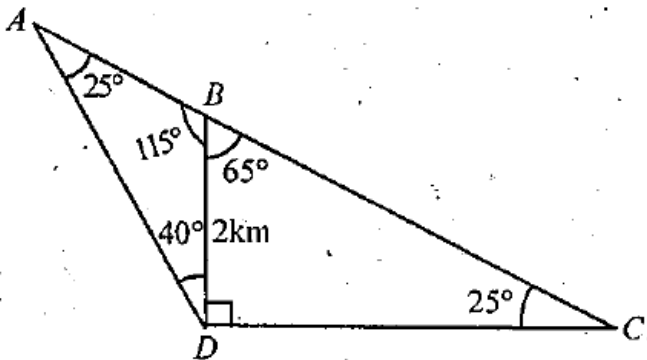
$$\therefore \cos \gamma = h^2 \cot^2 \alpha + h^2 \cot^2 \beta - d^2 / 2 h^2 \cot \alpha \cot \beta$$

$$\therefore -2h^2 \cot \alpha \cot \beta \cos \gamma + h^2 \cot^2 \alpha + h^2 \cot^2 \beta = d^2$$

$$\Rightarrow h = d / \sqrt{\cot^2 \alpha + \cot^2 \beta - 2 \cot \alpha \cot \beta \cos \gamma}$$

Sol 9.

According to question figure is as follows



Here, $\angle BDC = 90^\circ$, $BD = 2 \text{ km}$

$$\angle BDA = 40^\circ \Rightarrow \angle ADC = 130^\circ$$

$$\therefore \angle DAC = 180^\circ - (25^\circ + 130^\circ) = 25^\circ$$

From the figure, in ΔABD , using sine law

$$AD / \sin 115^\circ = BD / \sin 25^\circ \Rightarrow$$

$$AD = 2 \sin (90^\circ + 25^\circ) / \sin 25^\circ = 2 \cos 25^\circ / \sin 25^\circ$$

$$\Rightarrow AD = 2 \cot 25^\circ = 2 \sqrt{1 / \sin^2 25^\circ - 1} = 2 \sqrt{1 / (0.423)^2 - 1} = 4.28 \text{ km}$$

Sol 10.

KEY CONCEPT:

Ex - radii of a ΔABC are $r_1 = \Delta / s - a$, $r_2 = \Delta / s - b$

$r_3 = \Delta / s - c$ As r_1, r_2, r_3 are in H. P. $\therefore 1/r_1, 1/r_2, 1/r_3$ are in AP

$\Rightarrow s - a/\Delta, s - b/\Delta, s - c/\Delta$ are in AP

$\Rightarrow s - a, s - b, s - c$ are in AP

$\Rightarrow -a, -b, -c$ are in AP.

$\Rightarrow a, b, c$ are in A. P.

Sol 11.

Given that in ΔABC

$$\cos A + \cos B + \cos C = 3/2$$

$$\Rightarrow b^2 + c^2 - a^2/2bc + a^2 + c^2 - b^2/2ac + a^2 + b^2 - c^2/2ab = 3/2$$

$$ab^2 + ac^2 - a^3 + a^2b + bc^2 - b^3 + ac^2 + b^2c - c^3 = 3abc$$

$$\Rightarrow ab^2 + ac^2 + bc^2 + ba^2 + ca^2 + cb^2 - 6abc = a^3 + b^3 + c^3 - 3abc$$

$$\Rightarrow a(b-c)^2 + b(c-a)^2 + c(a-b)^2 = (a+b+c/2) [(a-b)^2 + (b-c)^2 + (c-a)^2]$$

$$\Rightarrow (a+b-c)(a-b)^2 + (b+c-a)(b-c)^2 + (c+a-b)(c-a)^2 = 0 \dots\dots\dots (1)$$

$$a + b > c$$

$b + c > a$ {sum of any two sides of a Δ is greater than the third side

As we know that $c + a > b$

\therefore Each part on the LHS of eq. (1) has +ve coeff. Multiplied by perfect square, each must be separately zero

$$\therefore a - b = 0; b - c = 0; c - a = 0 \Rightarrow a = b = c$$

Hence Δ is an equilateral Δ

ALTERNATE SOLUTION:

Given that $\cos A + \cos B + \cos C = 3/2$ in ΔABC

$$\Rightarrow 2 \cos A + B/2 \cos A - B/2 = 3/2 - \cos C$$

$$\Rightarrow 2 \sin C/2 \cos A - B/2 = 3 - 2 \cos C/2$$

$$\Rightarrow 2 \sin C/2 \cos A - B/2 = 3 - 2(1 - 2 \sin^2 C/2/4 \sin C/2)$$

$$\Rightarrow \cos(A - B/2) = 1 + 4 \sin^2 C/2 / 4 \sin C/2$$

$$\Rightarrow \cos(A - B/2) = 1 + 4 \sin^2 C/2 - 4 \sin C/2 + 4 \sin C/2 / 4 \sin C/2$$

$$\Rightarrow \cos(A - B/2) = (1 - 2 \sin C/2)^2/4 \sin C/2 + 1$$

Which is possible only when

$$1 - 2 \sin C/2 = 0 \Rightarrow \sin C/2 = 1/2$$

$$\Rightarrow C/2 = 30^\circ \Rightarrow C = 60^\circ$$

$$\text{Also then } \cos A - B/2 = 1 \Rightarrow A - B/2 = 0 \Rightarrow A - B = 0 \dots\dots\dots (1)$$

$$\text{And } A + B = 180^\circ - 60^\circ = 120^\circ \Rightarrow A + B = 120^\circ \dots\dots\dots (2)$$

From (1) and (2) $A = B = 60^\circ$

Thus we get $A = B = C = 60^\circ \quad \therefore \Delta ABC$ is an equilateral Δ .

Sol 12.

Given that, in ΔABC ,

$$b + c/11 = c + a/12 = a + b/13$$

Where a, b, c are the lengths of sides, BC, CA and AB respectively.

$$\text{Let } b + c/11 = c + a/12 = a + b/13 = k$$

$$\Rightarrow b + c = 11k \quad \dots\dots\dots (1)$$

$$\Rightarrow c + a = 11k \quad \dots\dots\dots (2)$$

$$\Rightarrow a + b = 13k \quad \dots\dots\dots (3)$$

Adding the above three eqs. We get

$$2(a + b + c) = 36k$$

$$\Rightarrow a + b + c = 18k \quad \dots\dots\dots (4)$$

Solving each of (1), (2) and (3) with (4), we get

$$\text{Now, } \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{36k^2 + 25k^2 - 49k^2/2 \times 6k \times 5k}{2 \times 6k \times 5k} = \frac{12k^2}{60k^2} = \frac{1}{5}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca} = \frac{25k^2 + 49k^2 - 36k^2/2 \times 5k \times 7k}{2 \times 5k \times 7k}$$

$$= \frac{38k^2}{70k^2} = \frac{19}{35}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{49k^2 + 36k^2 - 25k^2/2 \times 7k \times 6k}{2 \times 7k \times 6k}$$

$$= \frac{60k^2}{84k^2} = \frac{5}{7}$$

$$\therefore \cos A / \frac{1}{5} = \cos B / \frac{19}{35} = \cos C / \frac{5}{7}$$

$$\Rightarrow \cos A / \frac{7}{35} = \cos B / \frac{19}{35} = \cos C / \frac{25}{35}$$

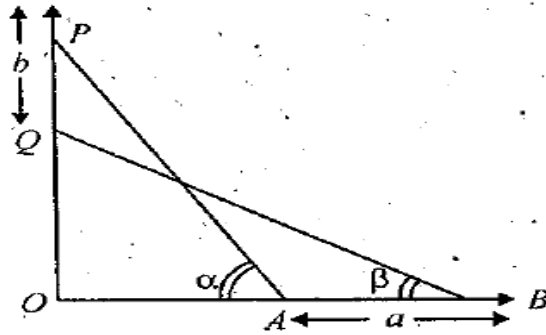
$$\Rightarrow \cos A/7 = \cos B/19 = \cos C/25$$

Sol 13.

Let the length of the ladder, then

$$\text{In } \Delta OQB, \cos \beta = \frac{OB}{BQ}$$

$$\Rightarrow OB = \ell \cos \beta \quad \dots\dots\dots (1)$$



Similarly in ΔOPA , $\cos \alpha = OA/PA$

$$\Rightarrow OA = \ell \cos \alpha \dots\dots\dots (2)$$

$$\text{Now } a = OB - OA = \ell (\cos \beta - \cos \alpha) \dots\dots\dots (3)$$

Also from ΔOAP , $OP = \ell \sin \alpha$

And in OQB ; $OQ = \ell \sin \beta$

$$\therefore b = OP - OQ = \ell (\sin \alpha - \sin \beta) \dots\dots\dots (4)$$

Dividing eq. (3) by (4) we get

$$a/b = \cos \beta - \cos \alpha / \sin \alpha - \sin \beta$$

$$\Rightarrow \frac{a}{b} = \frac{2 \sin \left(\frac{\alpha + \beta}{2} \right) \sin \left(\frac{\alpha - \beta}{2} \right)}{2 \cos \left(\frac{\alpha + \beta}{2} \right) - \sin \left(\frac{\alpha - \beta}{2} \right)}$$

$$\Rightarrow a/b = \tan (\alpha + \beta/2)$$

Thus , $a = b \tan (\alpha + \beta/2)$ is proved

Sol 14.

Let AD be the median in ΔABC .

Let $\angle B = \theta$ then $\angle C = 105^\circ - \theta$

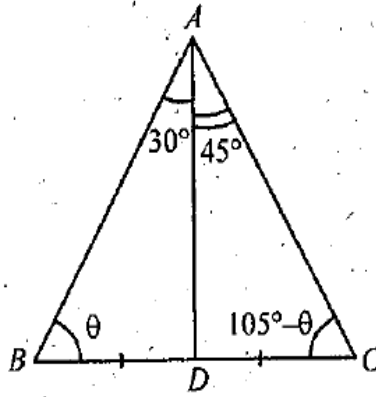
In ΔABD , using sine law, we get

$$BD/\sin 30^\circ = AD/\sin 3 \theta \Rightarrow BD = AD/2 \sin \theta$$

In ΔACD , using sine law, we get

$$DC/\sin 45^\circ = AD/\sin (105^\circ - \theta) \Rightarrow DC = AD/\sqrt{2} \sin(105^\circ - \theta)$$

As $BD = DC$



$$\Rightarrow AD/2 \sin \theta = AD/\sqrt{2} \sin (105^\circ - \theta)$$

$$\Rightarrow \sin (90^\circ + 15^\circ - \theta) = \sqrt{2} \sin \theta$$

$$\Rightarrow \cos 15^\circ \cos \theta + \sin 15^\circ \sin \theta = \sqrt{2} \sin \theta$$

$$\Rightarrow \cot \theta = \sqrt{2} - \sin 15^\circ / \cos 15^\circ = 5 - \sqrt{3} / \sqrt{3} + 1 = 3\sqrt{3} - 4$$

$$\Rightarrow \operatorname{cosec} \theta = \sqrt{1 + \cot^2 \theta} = \sqrt{1 + 27 + 16 - 24\sqrt{3}}$$

$$\Rightarrow \operatorname{cosec} \theta = 2\sqrt{11} - 6\sqrt{3}$$

$$\therefore BD = AD/2 \sin \theta = 1/\sqrt{11 - 6\sqrt{3}} \times 2\sqrt{11 - 6\sqrt{3}}/2 = 1$$

$$\therefore BC = 2 BD = 2 \text{ units}$$

Sol 15.

We are given that in ΔABC $\cos A \cos B + \sin A \sin B \sin C = 1$

$$\Rightarrow \sin A \sin B \sin C = 1 - \cos A \cos B$$

$$\Rightarrow \sin C = 1 - \cos A \cos B / \sin A \sin B$$

$$\Rightarrow 1 - \cos A \cos B / \sin A \sin B \leq 1 \quad [\because \sin C \leq 1]$$

$$\Rightarrow 1 - \cos A \cos B \leq \sin A \sin B$$

$$\Rightarrow 1 \leq \cos A \cos B + \sin A \sin B$$

$$\Rightarrow 1 \leq \cos(A - B)$$

$$\Rightarrow 1 \leq \cos(A - B)$$

But we know $\cos(A - B) \leq 1$

$$\therefore \text{We must have } \cos(A - B) = 1$$

$$\Rightarrow A - B = 0$$

$$\Rightarrow A = B$$

$$\therefore \cos A \cos A + \sin A \sin A \sin C = 1 \quad [\text{For } A = B]$$

$$\Rightarrow \cos^2 A + \sin^2 A \sin C = 1$$

$$\Rightarrow \sin^2 A \sin C = 1 - \cos^2 A$$

$$\Rightarrow \sin^2 A \sin C = \sin^2 A$$

$$\Rightarrow \sin^2 A (\sin C - 1) = 0$$

$$\Rightarrow \sin A = 0 \text{ or } \sin C = 1$$

The only possibility is $\sin C = 1 \Rightarrow C = \pi/2$

$$\therefore A + B = \pi/2$$

$$\text{But } A = B \Rightarrow A = B = \pi/4$$

\therefore By Sine law in ΔABC ,

$$a/\sin A = b/\sin B = c/\sin C$$

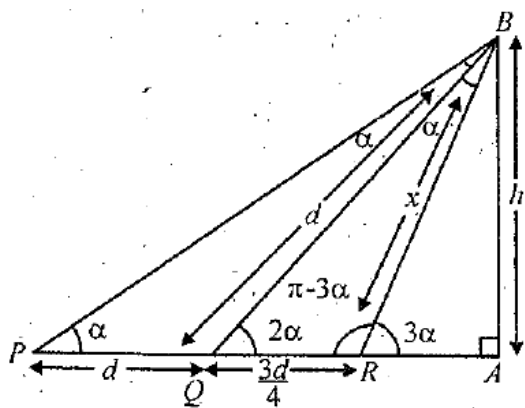
$$\Rightarrow a/\sin 45^\circ = b/\sin 45^\circ = c/\sin 90^\circ$$

$$\Rightarrow a/1/\sqrt{2} = b/1/\sqrt{2} = c/1$$

$$\Rightarrow a/1 = b/1 = 1/\sqrt{2} \Rightarrow a : b : c = 1 : 1 : \sqrt{2}$$

Hence proved the result.

Sol 16.



Let $RB = x$

$\angle BQR$ is ext \angle of ΔPBQ

$$\therefore \angle PBQ = 2\alpha - \alpha = \alpha$$

Now in ΔPBQ , $\angle PBQ = \angle QPB$

$$\Rightarrow PQ = QB = d$$

Also $\angle BRA$ is ext. \angle of ΔBQR

$$\therefore \angle QBR = 3\alpha - 2\alpha = \alpha$$

And $\angle BRQ = \pi - 3\alpha$ (linear pair)

Now in ΔBQR , by applying Sine Law, we get

$$d/\sin(\pi - 3\alpha) = 3d/4 / \sin \alpha = x/\sin 2\alpha$$

$$\Rightarrow d/\sin 3\alpha = 3d/4 \sin \alpha = x/\sin^2 \alpha$$

$$\Rightarrow d/3 \sin \alpha - 4 \sin^3 \alpha = 3d/4 \sin \alpha = x/2 \sin \alpha \cos \alpha$$

$$\Rightarrow d/3 - 4 \sin^2 \alpha = 3d/4 = x/2 \cos \alpha \dots\dots\dots (I)$$

$$(I) \qquad (II) \qquad (III)$$

From eq. (I), I = II

$$\Rightarrow d/3 - 4 \sin^2 \alpha = 3d/4 \Rightarrow 4 = 9 - 12 \sin^2 \alpha$$

$$\Rightarrow \sin^2 \alpha = 5/12 \Rightarrow \cos^2 \alpha = 7/12$$

Also from eq. (I) using (II) and (III), we have

$$3d/4 = x/2 \cos \alpha \Rightarrow 4x^2 = 9 d^2 \cos^2 \alpha$$

$$x^2 = 9d^2/4 = 7/12 = 21/16 d^2 \dots\dots\dots (3)$$

Again from ΔABR , we have $\sin 3\alpha = h/x$

$$\Rightarrow 3 \sin \alpha - 4 \sin^3 \alpha = h/x \Rightarrow \sin \alpha(3 - 4 \sin^2 \alpha) = h/x$$

$$\Rightarrow \sin \alpha [3 - 4 \times 5/12] = h/x \quad (\text{using } \sin^2 \alpha = 5/12)$$

$$\Rightarrow 4/3 \sin \alpha = h/x$$

Squaring both sides, we get

$$16/9 \sin^2 \alpha = h^2/x^2 \quad 16/9 \times 2/12 = h^2/x^2$$

(again using $\sin^2 \alpha = 5/12$)

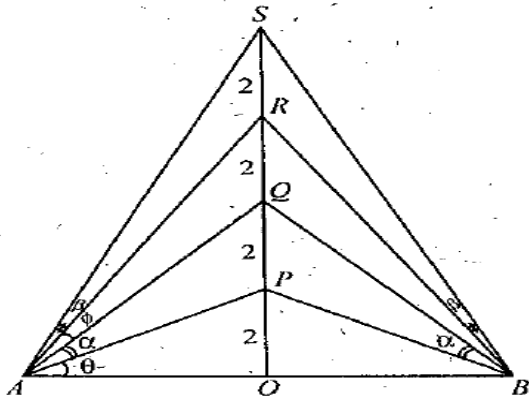
$$\Rightarrow h^2 = 4 \times 5/9 \times 3 \quad x^2 \Rightarrow h^2 = 20/27 \times 21/16 d^2$$

[using value of x^2 from eq. (3)]

$$\Rightarrow h^2 = 35/36 d^2 \Rightarrow 36 h^2 = 35 d^2$$

Which was to be proved.

Sol 17.



Let O be the mid point of AB = 8 m

$$\therefore OA = OB = 4 \text{ m}$$

Also OP = 2 m is the initial position of the object which is 2m long. Also we are given, $ds/dt = 2t + 1$
Integrating we get, $s = t^2 + t + k$ (where s is the distance of top of object from O)

When $t = 0, s = OP = 2$

$$\therefore k = 2$$

$$\therefore s = t^2 + t + 2 \dots\dots\dots (1)$$

For $t = 1, s = 4 = OQ \therefore PQ = 2$ where PQ is the position of object after 1 sec.

For $t = 2, s = 8 = OS$ but $RS = 2$ where RS is the position of the object after 2 sec.

$$\therefore OR = OS - RS = 6$$

Also, $OQ = 4$

$$\therefore QR = OR - OQ = 6 - 4 = 2$$

$$\therefore OP = PQ = QR = RS = 2$$

As per the condition of the question, PQ and RS, the position of the object at $t = 1$ and $t = 2$ subtend angles α and β at A and B respectively.

Now let $\angle PAO = \theta$ so that $\tan \theta = 2/4 = 1/2$

Also $\tan (\alpha + \theta) = 4/4 = 1$

Now, $\tan \alpha = \tan [(\alpha + \theta) - \theta]$

$$= \frac{\tan (\alpha + \theta) - \tan \theta}{1 + \tan (\alpha + \theta) \tan \theta} = \frac{1 - \frac{1}{2}}{1 + \frac{1}{2}} = 1/3$$

$$\therefore \sin \alpha = 1/\sqrt{10} \text{ and } \cos \alpha = 3/\sqrt{10} \dots\dots\dots (2)$$

Similarly taking $\angle RAQ = \phi$ so that

$$\tan \beta = \tan [(\theta + \alpha + \phi + \beta) - (\theta + \alpha + \beta)]$$

$$= \frac{\tan(\theta + \alpha + \phi + \beta) - \tan(\theta + \alpha + \beta)}{1 + \tan(\theta + \alpha + \phi + \beta) \cdot \tan(\theta + \alpha + \beta)}$$

$$= \frac{\frac{8}{4} - \frac{6}{4}}{1 + \frac{8}{4} \cdot \frac{6}{4}} = \frac{2 - 3/2}{1 + 2(3/2)} = 1/8$$

$$\therefore \sin \beta = 1/\sqrt{65} \text{ and } \cos \beta = 8/\sqrt{65} \dots\dots\dots (3)$$

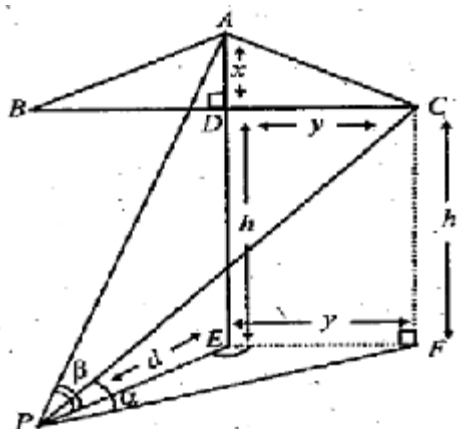
$\therefore \cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$ Using equations (2) and (3) we get

$$\cos(\alpha - \beta) = 3/\sqrt{10} \cdot 8/\sqrt{65} + 1/\sqrt{10} \cdot 1/\sqrt{65}$$

$$= 25/5 \sqrt{2} \sqrt{13} = 5/\sqrt{26}$$

Sol 18.

Let ABC be the isosceles triangular sign board with BC horizontal. DE be the pole of height h. Let the man be standing at P such that PE = d



Also $\angle APE = \beta$

$\angle CPF = \alpha$

Let $AD = x$ be altitude of ΔABC .

As ΔABC is isosceles with $AB = AC$

$\therefore D$ is mid point of BC .

Hence $BC = 2y$.

Now in ΔAPE ,

$$\tan \beta = \frac{h + x}{d} \Rightarrow x = d \tan \beta - h \dots\dots\dots (1)$$

$$\text{In } \Delta CPE, \tan \alpha = \frac{h}{PF} \Rightarrow \tan \alpha = \frac{h}{\sqrt{d^2 + y^2}}$$

$$\Rightarrow y^2 + d^2 = h^2 \cot^2 \alpha$$

$$\Rightarrow y = \sqrt{h^2 \cot^2 \alpha - d^2} \dots\dots\dots (2)$$

Now area of $\Delta ABC = 1/2 \times BC \times AD$

$$= 1/2 \times 2y \times x = xy$$

$$= (d \tan \beta - h) \sqrt{h^2 \cot^2 \alpha - d^2} \text{ [Using (1) and (2)]}$$

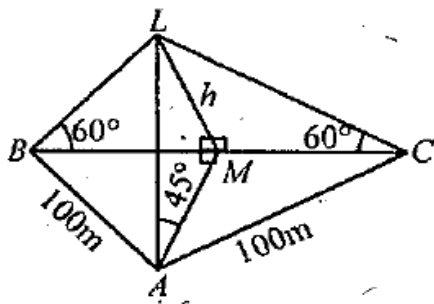
Sol 19.

Let ABC be the triangle region with $AB = AC = 100\text{m}$ Let M be the mid point of BC at which tower LM stands.

As ΔABC is isosceles and M is mid pt. of BC

$\therefore AM \perp BC$.

Let $LM = h$ be the ht. of tower.



In ΔALM , $\tan 45^\circ = LM/MA \Rightarrow LM = MA$

$\therefore MA = h$

Also in ΔBLM , $\tan 60^\circ = LM/BM$

$$\Rightarrow \sqrt{3} = h/BM \Rightarrow BM = h/\sqrt{3}$$

Now in rt ΔAMB , we have, $AB^2 = AM^2 + BM^2$

$$\Rightarrow (100)^2 = h^2 + h^2/3$$

$$\Rightarrow 4h^2/3 = 10000$$

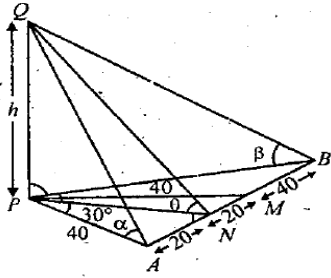
$$\Rightarrow h = 50\sqrt{3} \text{ m.}$$

Sol 20.

Let $PQ = h$

As A and B are located to the south and east of P respectively,

$\therefore \angle APB = 90^\circ$. M is mid pt of AB. PAM is an equilateral Δ



$\therefore \angle APM = 60^\circ :$

Also $PN \perp AB$, therefore $AN = NM = 20$ m

$\Rightarrow AP = 40$ m

Let angles of elevation of top of the tower from A, N and B be α , θ and β respectively. At Q, $\tan \theta = 2$

In ΔPQN $\tan \theta = PQ/PN$

$\Rightarrow 2 = h/PN \Rightarrow PN = h/2 \dots\dots\dots (1)$

Also in ΔAPM , $\angle APM = 60^\circ$ (being equilateral Δ) and PN is altitude $\therefore \angle APN = 30^\circ$ (as in equilateral Δ altitude bisects the vertical angle).

\therefore In ΔAPN $\tan \angle APN = AN/PN$

$\Rightarrow \tan 30^\circ = 20 / h/2$ [Using eq. (1)]

$\Rightarrow h/2\sqrt{3} = 20 \Rightarrow h = 40\sqrt{3}$ m.

In ΔAPQ $\tan \alpha = h/AP \Rightarrow \tan \alpha = 40\sqrt{3}/40 = \sqrt{3}$

$\Rightarrow \alpha = 60^\circ$ Also in ΔABQ $\tan \beta = h/PB$ but in rt ΔPNB

$PB = \sqrt{PN^2 + NB^2} = \sqrt{(20\sqrt{3})^2 + (60)^2}$

$\therefore PB = \sqrt{1200 + 3600} = \sqrt{4800} = 40\sqrt{3}$

$\therefore \tan \beta = 40\sqrt{3}/40\sqrt{3} \Rightarrow \tan \beta = 1 \Rightarrow \beta = 45^\circ$

Thus $h = 40\sqrt{3}$ m ; \angle 's of elevation are $60^\circ, 45^\circ$

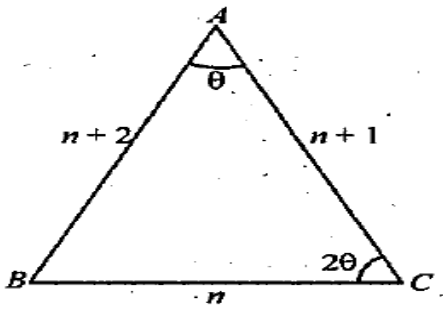
Sol 21.

Let the sides of Δ be $n, n + 1, n + 2$ where $n \in \mathbb{N}$.

Let $a = n, b = n + 1, c = n + 2$

Let the smallest, angle $\angle A = \theta$ then the greatest $\angle C = 2\theta$ In ΔABC by applying Sine Law we get,

$\sin \theta/n = \sin 2\theta/n + 2$



$$\Rightarrow \sin \theta/n = 2 \sin \theta \cos \theta/n + 2 \Rightarrow 1/n = 2 \cos \theta/n + 2 \text{ (as } \sin \theta \neq 0)$$

$$\Rightarrow \cos \theta = n + 2/2n \dots\dots\dots (1)$$

In ΔABC by Cosine Law, we get

$$\cos \theta = (n + 1)^2 + (n + 2)^2 - n^2/2(n + 1)(n + 2) \dots\dots\dots (2)$$

Comparing the values of $\cos \theta$ from (1) and (2), we get

$$(n + 1)^2 + (n + 2)^2 - n^2/2(n + 1)(n + 2) = n + 2/2n$$

$$\Rightarrow (n + 2)^2 (n + 1) = n(n + 2)^2 + n(n + 1)^2 - n^3$$

$$\Rightarrow n (n + 2)^2 (n + 2) = n(n + 2)^2 + n(n + 1)^2 - n^3$$

$$\Rightarrow n^2 + 4n + 4 = n^3 + 2n^2 + n - n^3$$

$$\Rightarrow n^2 - 3n - 4 = 0 \Rightarrow (n + 1)(n - 4) = 0$$

$$\Rightarrow n = 4 \text{ (as } n \neq -1)$$

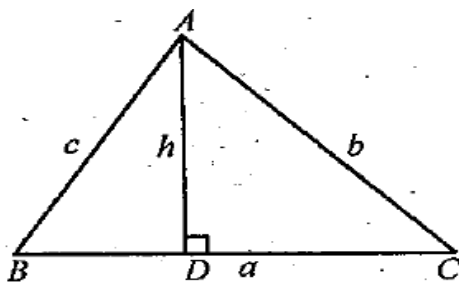
\therefore Sides of Δ are 4, 4 + 1, 4 + 2, i.e. 4, 5, 6.

Sol 22.

Given that, In ΔABC , base = a And $c/b = r$ To find altitude, h.

We have, in ΔABD ,

$$h = c \sin B = c a \sin B/a$$



$$= c k \sin A \sin B/k \sin A = c k \sin A \sin B/\sin (B + C)$$

$$= c \sin A \sin B \sin (B - C)/\sin (B + C) \sin (B - C) = c \sin A \sin B \sin (B - C)/\sin^2 B - \sin^2 C$$

$$= \frac{c \cdot \frac{a}{k} \cdot \frac{b}{k} \sin(B-C)}{\frac{b^2}{k^2} - \frac{c^2}{k^2}} = abc \sin(B-C) / (b^2 - c^2)$$

$$= a(c/b) \sin(B-C) / (1 - (c/b)^2) = ar \sin(B-C) / (1 - r^2) \leq ar / (1 - r^2)$$

[∵ sin(B - C) ≤ 1] ∴ h ≤ ar / (1 - r^2) Hence Proved.

Sol 23.

Let the man initially be standing at 'A' and 'B' be the position after walking a distance 'c', so total Distance becomes 2c and the objects being observed are at 'C' and 'D'.



Now we have OA = c, AB = 2c

Let CO = x and CD = d

Let ∠CAD = α and ∠CBD = β

∠ACO = θ and ∠ADC = φ

∠BCD = ψ and ∠BCO = θ₁

In Δ ACO, tan θ = AO/CO ⇒ tan θ = c/x (1)

In Δ ADO, tan φ = c/(x + d) (2)

Now, θ = α + φ (Using ext. ∠ thm.)

$$\Rightarrow \alpha = \theta - \phi \Rightarrow \tan \alpha = \tan(\theta - \phi) \Rightarrow \frac{\tan \theta - \tan \phi}{1 + \tan \theta \tan \phi}$$

$$= \frac{c/x - c/(x + d)}{1 + c/d \cdot c/x + d} \quad \text{(using equations (1) and (2))}$$

$$\Rightarrow \tan \alpha = \frac{cx + cd - cx/x^2 + dx + c^2}{x^2 + c^2 + xd}$$

$$\Rightarrow x^2 + c^2 + xd = cd \cot \alpha \quad \dots \dots \dots (3)$$

Again in Δ ADO

$$\tan \psi = \frac{3c}{x + d} \quad \dots \dots \dots (4)$$

$$\tan \theta_1 = \frac{3c}{x} \quad \dots \dots \dots (5)$$

But θ₁ = ψ + β (by text ∠ thrm)

$$\Rightarrow \beta = \theta_1 - \psi$$

$$\Rightarrow \tan \beta = \tan (\theta_1 - \psi) = \frac{\tan \theta_1 - \tan \psi}{1 + \tan \theta_1 \tan \psi}$$

$$\Rightarrow \tan \beta = \frac{\frac{3c}{x} - \frac{3c}{x+d}}{1 + \frac{3c}{x} \cdot \frac{3c}{x+d}} \quad [\text{Using (4) and (5)}]$$

$$\Rightarrow \tan \beta = \frac{3cd}{x^2 + xd + 9c^2}$$

$$\Rightarrow x^2 + xd + 9c^2 = 3cd \cot \beta \dots\dots\dots (6)$$

From (3) and (6), we get

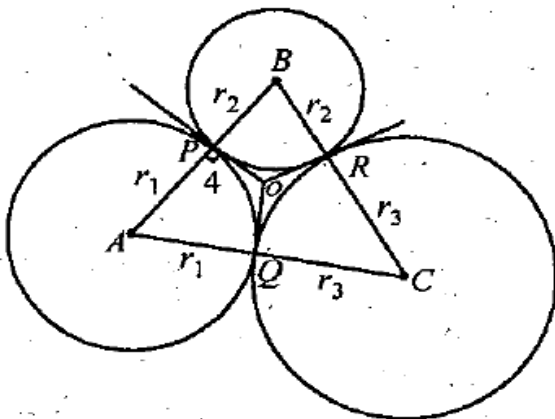
$$8c^2 = 3cd \cot \beta - cd \cot \alpha$$

$$\Rightarrow d = \frac{8c}{3} \cot \beta - \cot \alpha \quad \text{Hence proved.}$$

Sol 24.

Let us consider circles with centers at A, B and C and with radii r_1, r_2 and r_3 respectively which touch each other externally at P, Q and R. Let the common tangents at P, Q and R meet each other at O. Then $OP = OQ = OR = 4$ (given)(lengths of tangents from a pt to a circle are equal).

Also $OP \perp AB, OQ \perp AC, OR \perp BC$.



$\Rightarrow O$ is the in centre of the ΔABC Thus for ΔABC

$$s = \frac{(r_1 + r_2) + (r_2 + r_3) + (r_3 + r_1)}{2}$$

$$\text{i.e. } s = (r_1 + r_2 + r_3)$$

$$\therefore \Delta = \sqrt{(r_1 + r_2 + r_3) r_1 r_2 r_3} \quad (\text{Heron's formula})$$

$$\text{Now } r = \Delta/s$$

NOTE THIS STEP :

$$\Rightarrow 4 = \frac{\sqrt{(r_1 + r_2 + r_3) r_1 r_2 r_3}}{r_1 + r_2 + r_3}$$

$$\Rightarrow 4 = \frac{\sqrt{r_1 r_2 r_3}}{\sqrt{r_1 + r_2 + r_3}}$$

$$\Rightarrow \frac{r_1 r_2 r_3}{r_1 + r_2 + r_3} = \frac{16}{1} \Rightarrow r_1 \cdot r_2 \cdot r_3 : r_1 + r_2 + r_3 = 16 : 1$$

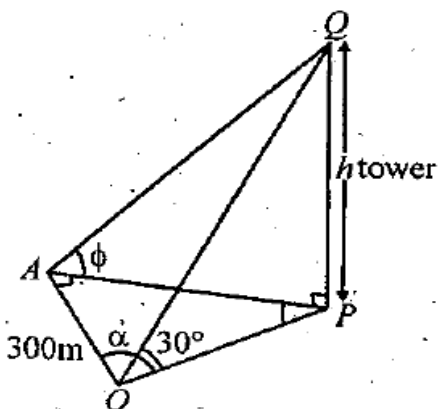
Sol 25.

Let PQ be the tower of height h. A is in the north of O and P is towards east of A.

$$\therefore \angle OAP = 90^\circ$$

$$\angle QOP = 30^\circ$$

$$\angle QAP = \phi$$



$$\angle AOP = \alpha \text{ s. t. } \tan \alpha = 1/\sqrt{2}$$

$$\text{Now in } \triangle OPQ, \tan 30^\circ = h/OP \Rightarrow OP = h/\sqrt{3} \dots\dots\dots (1)$$

$$\text{In } \triangle APQ, \tan \phi = h/AP \Rightarrow AP = h \cot \phi \dots\dots\dots (2)$$

Given that,

$$\tan \alpha = 1/\sqrt{2} \Rightarrow \sin \alpha = \tan \alpha / \sqrt{1 + \tan^2 \alpha} = 1/3$$

$$\text{Now in } \triangle AOP, \sin \alpha = AP/OP \Rightarrow 1/\sqrt{3} = h \cot \phi / h \sqrt{3} \text{ [Using (1) and (2)]}$$

$$\Rightarrow \cot \phi = 1$$

$$\Rightarrow \phi = 45^\circ$$

Again in $\triangle OAP$, using Pythagoras theorem, we get

$$OP^2 = OA^2 + AP^2$$

$$\Rightarrow 3h^2 = 90000 + h^2 \cot^2 45^\circ$$

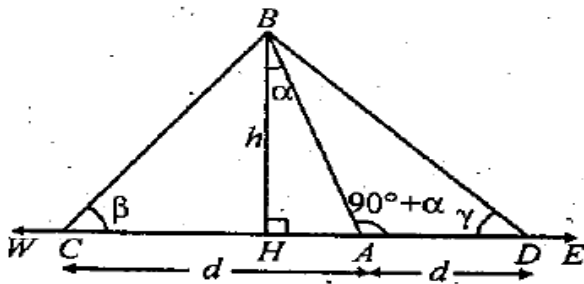
$$\Rightarrow h = 150 \sqrt{2} \text{ m}$$

Sol 26.

Let AB be the tower leaning towards west making an angle α with vertical

At C, \angle of elevation of B is β and at D the

\angle of elevation of B is γ , $CA = AD = d$



When in ΔABH

$$\Rightarrow \tan \alpha = AH/h \Rightarrow AH = h \tan \alpha \dots\dots\dots (1)$$

In ΔBCH , $\tan \beta = h/CH \Rightarrow CH = h \cot \beta$

$$\Rightarrow d - AH = h \cot \beta$$

$$\Rightarrow d = h (\tan \alpha + \cot \beta) \dots\dots\dots (2)$$

(Using eqⁿ (1))

In ΔBDH , $\tan \gamma = BH/HD \Rightarrow h/AH + d$

$$\Rightarrow AH + d = h \cot \gamma$$

$$\Rightarrow d = h (\cot \gamma - \tan \alpha) \dots\dots\dots (3)$$

(Using eqⁿ (1))

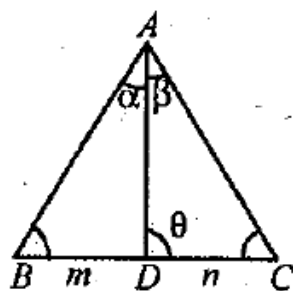
Comparing the values of d from (2) and (3), we get

$$h(\tan \alpha + \cot \beta) = h (\cot \gamma - \tan \alpha)$$

$$\Rightarrow 2 \tan \alpha = \cot \gamma - \cot \beta \text{ Hence Proved}$$

ALTERNATE SOLUTION :

KEY CONCEPT:



$m : n$ theorem : In ΔABC where point D divides BC in the ratio $m : n$ and $\angle ADC = \theta$

$$(i) (m + n) \cot \theta = n \cot B - m \cot C$$

(ii) $(m + n) \cot \theta = m \cot \alpha - n - \cot \beta$

In ΔBCD , A divides CD in the ratio 1 : 1 where base \angle 's are β and γ and $\angle BAD = 90^\circ + \alpha$

\therefore By applying m : n theorem we get

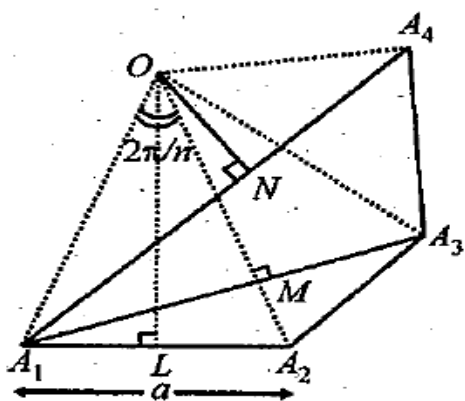
$(1 + 1) \cot (90^\circ + \alpha) = 1 \cdot \cot \beta - 1 \cdot \cot \gamma$

$\Rightarrow -2 \tan \alpha = \cot \beta - \cot \gamma \Rightarrow 2 \tan \alpha = \cot \gamma - \cot \beta$ Hence Proved.

Sol 27.

Let a be the side of n sided regular polygon

$A_1 A_2 A_3 A_4 \dots A_n$



$\therefore \angle$ Subtended by each side at centre will be

$= 2\pi/n$

Let $OL \perp A_1 A_2$

Then $\angle OLA_1 = 90^\circ, \angle A_1 OL = \pi/n$ ($\because OA_1 = OA_2$)

\therefore In $\Delta OA_1 L, \sin A_1 OL = A_1 L/O A_1$

$\Rightarrow \sin \pi/n = a/2 / O A_1$

$\Rightarrow O A_1 = a/2 \sin \pi/n \dots \dots \dots (1)$

Again by geometry it can be proved that $OM \perp A_1 A_3$

In $\Delta A_1 M, \sin 2\pi/n = A_1 M/O A_1$

$\Rightarrow A_1 M = O A_1 \sin 2\pi/n$

$\Rightarrow A_1 A_3 = 2a \sin 2\pi/n / 2 \sin \pi/n$ [Using eqⁿ (1)]

Also if $ON \perp A_1 A_4$, then ON bisects angle

$\angle A_1 O A_4 = 3(2\pi/n)$

$$\therefore \angle A_1 ON = 3\pi/n$$

$$\therefore \text{In } \Delta OA_1 N, \sin 3\pi/2 = A_1 N/OA_1$$

$$\Rightarrow A_1 N = OA_1 \sin 3\pi/n$$

$$\Rightarrow A_1 N_4 = 2A_1 N = 2a \sin 3\pi/n / 2 \sin \pi/n$$

But given that

$$1/A_1 A_2 = 1/A_1 A_3 + 1/A_1 A_4$$

$$\Rightarrow 1/a = 1/a \sin 2\pi/n / \sin \pi/n + 1/a \sin 3\pi/n / \sin \pi/n$$

$$\Rightarrow \sin 3\pi/n \sin 2\pi/n = (\sin 3\pi/n + \sin 2\pi/n) \sin \pi/n$$

$$\Rightarrow 2 \sin 3\pi/n \sin 2\pi/n = 2 \sin 3\pi/n + 2 \sin 2\pi/n \sin \pi/n$$

$$\Rightarrow \cos \pi/n - \cos 5\pi/n = \cos 2\pi/n - \cos 4\pi/n + \cos \pi/n - \cos 3\pi/n$$

$$\Rightarrow \cos 2\pi/n + \cos 5\pi/n = \cos 4\pi/n + \cos 3\pi/n$$

$$\Rightarrow 2 \cos 7\pi/2n \cos 3\pi/2n = 2 \cos 7\pi/2n \cos \pi/2n$$

$$\Rightarrow \cos 7\pi/2n (\cos 3\pi/2n - \cos \pi/2n) = 0$$

$$\Rightarrow \cos 7\pi/2n \cdot 2 \sin 2\pi/n \sin \pi/n = 0$$

$$\Rightarrow \cos 7\pi/2n = 0$$

$$\text{Or } \sin 2\pi/n = 0 \text{ or } \sin \pi/n = 0$$

$$\Rightarrow 7\pi/2n = (2k+1)\pi/2 \text{ or } 2\pi/n = k\pi/n \text{ or } \pi/n = p\pi$$

$$\Rightarrow n = 7/2k + 1 \text{ or } n = 2/k \text{ or } n = 1/k$$

But n should be a +ve integer being no. of sides and $n \geq 4$ (four vertices being considered in the question)

$$\therefore \text{the only possibility is } n = 7/2k + 1 \text{ for } k = 0$$

$$\therefore n = 7$$

Sol 28.

(I) a, b, c and Δ are rational.

$$\Rightarrow s = a + b + c/2 \text{ is also rational}$$

$$\Rightarrow \tan B/2 = \sqrt{(s-a)(s-c)/s(s-b)} = \Delta/s(s-b) \text{ is also rational}$$

$$\text{And } \tan C/2 = \sqrt{(s-a)(s-b)/s(s-c)} = \Delta/s(s-c) \text{ is also rational}$$

Hence (I) \Rightarrow (II).

(II) a, $\tan B/2$, $\tan C/2$ are rational.

$$\Rightarrow \sin B = \frac{2 \tan B/2}{1 + \tan^2 B/2}$$

And $\sin C = \frac{2 \tan C/2}{1 + \tan^2 C/2}$ are rational

$$\text{And } \tan A/2 = \tan [90^\circ - (B/2 + C/2)] = \cot (B/2 + C/2)$$

$$= \frac{1}{\tan (B/2 + C/2)} = \frac{1 - \tan B/2 \tan C/2}{\tan B/2 + \tan C/2} \text{ is rational}$$

$\therefore \sin A = \frac{2 \tan A/2}{1 + \tan^2 A/2}$ is rational.

Hence (II) \Rightarrow (III)

(III) a, $\sin A$, $\sin B$, $\sin C$ are rational.

$$\text{But } a/\sin A = 2R$$

$\Rightarrow R$ is rational

$$\therefore b = 2R \sin B, c = 2R \sin C \text{ are rational.}$$

$$\therefore \Delta = \frac{1}{2} bc \sin A \text{ is rational}$$

Hence (III) \Rightarrow (I).

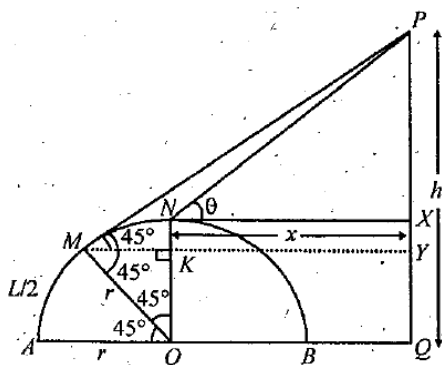
Sol 29.

Let AMNB be the semicircular arc of length $2L$, and PQ be the vertical tower so that A, B, Q are in the same horizontal line.

Let M be the pt. on arc s. t. $AB = L/2$ then as $AM = 1/4 AB$ we should have $\angle AOM = 45^\circ$, As at M the man just sees the top most pt P of the tower, tangent through M must pass through P and hence $\angle OMP = 90^\circ$ and then by simple geometry we get $\angle PMY = 45^\circ$.

Also N is the top most pt. of arc AB, hence ON must be vertical.

$$\therefore ON = r = XQ$$



$$\therefore PX = h - r, \text{ where } h \text{ is ht of tower } PQ, \text{ and } OK = YQ = OM \cos 45^\circ = r/\sqrt{2}$$

$$\text{Similarly, } MK = OM \sin 45^\circ = r/\sqrt{2}$$

∴ In ΔPMY we get $\tan 45^\circ = PY/MY$

$$\Rightarrow PY = MY$$

$$\Rightarrow h - QY = MK + KY$$

$$\Rightarrow h - OK = r/\sqrt{2} + x$$

$$\Rightarrow h - r/\sqrt{2} + r/\sqrt{2} + x$$

$$\Rightarrow x = h - r\sqrt{2} \dots\dots\dots (1)$$

In ΔPNX , $\tan \theta = PX/NX$

$$\Rightarrow \tan \theta = (h - r)/x$$

$$\Rightarrow x = (h - r) \cot \theta \dots\dots\dots (2)$$

Comparing the values of x from (1) and (2), we get

$$h - r\sqrt{2} = h \cot \theta - r \cot \theta$$

$$h = r(\sqrt{2} - \cot \theta)/(1 - \cot \theta)$$

$$\text{But arc length} = 2L = \pi r \Rightarrow r = 2L/\pi$$

$$\therefore h = 2L/\pi [\sqrt{2} - \cot \theta/1 - \cot \theta]$$

Sol 30.

Given that A, B, C , are three \angle 's of a Δ therefore

$$A + B + C = \pi \text{ Also } A = \pi/4 \Rightarrow B + C = 3\pi/4 \Rightarrow 0 < B, C < 3\pi/4$$

Now $\tan B \tan C = p$

$$\Rightarrow \sin B \sin C / \cos B \cos C = p/1$$

Applying componendo and dividendo, we get

$$\frac{\sin B \sin C + \cos B \cos C}{\cos B \cos C} - \frac{\sin B \sin C}{\cos B \cos C} = \frac{1+p}{1-p}$$

$$\Rightarrow \frac{\cos(B - C)}{\cos(B + C)} = \frac{1+p}{1-p}$$

$$\Rightarrow \cos(B - C) = \frac{1+p}{1-p} \left(-\frac{1}{\sqrt{2}}\right) \dots\dots\dots (1) \quad [\because B + C = 3\pi/4]$$

Now, as B and C can vary from 0 to $3\pi/4$

$$\therefore 0 \leq B - C < 3\pi/4$$

$$\Rightarrow \frac{1}{\sqrt{2}} < \cos(B - C) \leq 1$$

From eq" (1) substituting the value of $\cos(B - C)$, we get

$$-1/\sqrt{2} < 1 + p/\sqrt{2}(p - 1) \geq 1$$

$$\Rightarrow -1/\sqrt{2} < 1 + p/\sqrt{2}(p - 1) \text{ and } 1 + p/\sqrt{2}(p - 1) \leq 1$$

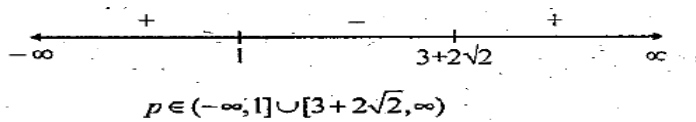
$$\Rightarrow 0 < 1 + p + 1/p - 1 \text{ and } (p + 1) - \sqrt{2}(p - 1)/\sqrt{2}(p - 1) \leq 0$$

$$\Rightarrow 2p/p - 1 > 0 \text{ and } p + 1 - \sqrt{2}p + \sqrt{2}/\sqrt{2}(p - 1) \leq 0$$

$$\Rightarrow p(p - 1) > 0 \text{ and } (1 - \sqrt{2})p + (\sqrt{2} + 1)/(p - 1) \leq 0$$

$$\Rightarrow p \in (-\infty, 0) \cup (1, \infty), \text{ and } -p + (\sqrt{2} + 1)^2/(p - 1) \leq 0$$

$$\Rightarrow [p - (3 + 2\sqrt{2})][p - 1] \geq 0$$



Combining the two cases, we get

$$p \in (-\infty, 0) \cup [3 + 2\sqrt{2}, \infty).$$

Sol 31.

Let A, B, C be the projections of the pts.

P, Q and M on the ground.

$$\angle POA = 60^\circ$$

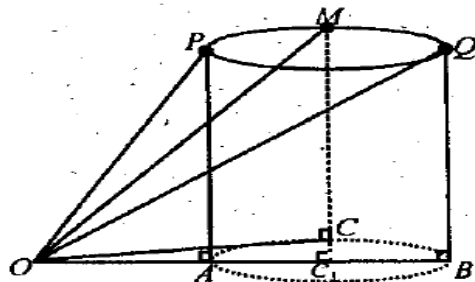
$$\angle QOB = 30^\circ$$

$$\angle MOC = \theta$$

Let h be the ht of circle from ground, then

$$AP = CM = BQ = h$$

Let OA = x and AB = d (diameter of the projection of the circle on ground with C₁ as centre).



$$\text{Now in } \Delta POA, \tan 60^\circ = h/x \Rightarrow x = h/\sqrt{3} \dots\dots\dots (1)$$

$$\text{In } \Delta QBO, \tan 30^\circ = h/x + d \Rightarrow x + d = h\sqrt{3}$$

$$\Rightarrow d = h\sqrt{3} - h/\sqrt{3} = 2h/\sqrt{3} \dots\dots\dots (2)$$

In ΔOMC , $\tan \theta = h/OC$

$$\Rightarrow \tan^2 \theta = h^2/OC^2 = h^2/OC^2 + C_1 C^2 = h^2/(x + d/2)^2 + (d/2)^2$$

$$= \frac{h^2}{\left(\frac{h}{\sqrt{3}} + \frac{h}{\sqrt{3}}\right)^2 + \left(\frac{h}{\sqrt{3}}\right)^2} \text{ [Using (1) and (2)]}$$

$$= \frac{h^2}{\frac{4h^2}{3} + \frac{h^2}{3}} = 3/5$$

Sol 32.

Let ABC is an equilateral Δ then

$$A = B = C = 60^\circ$$

$$\Rightarrow \tan A + \tan B \tan C = 3\sqrt{3}$$

Conversely, suppose

$$\tan A + \tan B + \tan C = 3\sqrt{3} \dots\dots\dots (1)$$

Now using A. M. \geq G. M. (equality occurs when no's are equal)

For $\tan A, \tan B, \tan C$, we get

$$\tan A + \tan B + \tan C/3 \geq (\tan A \tan B \tan C)^{1/3}$$

NOTE THIS STEP :

But in any ΔABC , know that

$$\tan A + \tan B \tan C = \tan A \tan B \tan C$$

\therefore Last inequality becomes

$$\tan A + \tan B + \tan C/3 \geq (\tan A + \tan B + \tan C)^{1/3}$$

$$\Rightarrow (\tan A + \tan B + \tan C)^{2/3} \geq 3$$

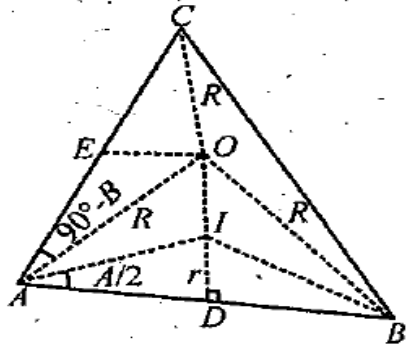
$$\Rightarrow \tan A + \tan B + \tan C \geq 3\sqrt{3}$$

Where equality occurs when $\tan A, \tan B, \tan C$ are equal, i.e. $A = B = C$

$\Rightarrow \Delta ABC$ is equilateral.

Sol 33.

In ΔABC , O and I are circumcentre and indenter of Δ respectively and R and r are the respective radii of circum circle and in circle.



To prove $(IO)^2 = R^2 - 2Rr$

First of all we will find IO. Using cosine law in ΔAOI

$$\cos \angle OAI = \frac{OA^2 + AI^2 - OI^2}{2 \cdot OA \cdot AI} \dots \dots \dots (1)$$

where $OA = R$

In ΔAID , $\sin A/2 = r/AI$

$$AI = r/\sin A/2$$

$$= 4R \sin B/2 \sin C/2 \text{ [Using } r = 4R \sin A/2 \sin B/2 \sin C/2 \text{]}$$

Also, $\angle OAI = \angle IAE - \angle OAE$

$$= A/2 - (90^\circ - \angle AOE)$$

$$= A/2 - 90^\circ + 1/2 \angle AOC$$

$$= A/2 - 90^\circ + 1/2 \cdot 2B \text{ (}\because O \text{ is circumcentre } \therefore \angle AOC = 2\angle B \text{)}$$

$$= A/2 + B - A + B + C/2$$

$$= B - C/2$$

Substituting all these values in equation (1) we get

$$\cos (B - C)/2 = \frac{R^2 + 16 R^2 \sin^2 B/2 \sin^2 C/2 - OI^2}{2 \cdot R \cdot 4 R \sin C/2}$$

$$\Rightarrow OI^2 = R^2 + 16R^2 \sin^2 B/2 \sin^2 C/2 - 8R^2 \sin B/2 \sin C/2 \cos B - C/2$$

$$= R^2 [1 + \sin B/2 \sin C/2 \{2 \sin B/2 \sin C/2 - \cos B - C/2 \}]$$

$$= R^2 [1 + 8 \sin B/2 \sin C/2 \{2 \sin B/2 \sin C/2 - \cos B/2 \cos C/2 - \sin B/2 \sin C/2 \}]$$

$$= R^2 [1 + 8 \sin C/2 \{ \sin B/2 \sin C/2 - \cos B/2 \cos C/2 \}]$$

$$= R^2 [1 - 8 \sin B/2 \sin C/2 \cos B + C/2]$$

$$= R^2 [1 - 8 \sin A/2 \sin B/2 \sin C/2]$$

$$= R^2 - 2R \cdot 4R \sin A/2 \sin B/2 \sin C/2$$

$$= R^2 - 2Rr. \text{ Hence Proved}$$

Again if ΔOIB is right \angle ed Δ then

$$\Rightarrow OB^2 = OI^2 + IB^2$$

$$\Rightarrow R^2 = R^2 - 2Rr + r^2/\sin^2 B/2$$

NOTE THIS STEP:

$$[\because \text{In } \Delta IBD \sin B/2 = r/IB]$$

$$\Rightarrow 2R \sin^2 B/2 = r$$

$$\Rightarrow 2abc/4\Delta (s-a)(s-c)/ac = \Delta/s$$

$$\Rightarrow b(s-a)(s-c) = 2(s-a)(s-b)(s-c)$$

$$\Rightarrow b = 2s - 2b \Rightarrow b = a + b - c$$

$$\Rightarrow a + c = 2b \Leftrightarrow a, b, c \text{ are in A. P.}$$

$$\Rightarrow b \text{ is A. M} > \text{ between } a \text{ and } c. \text{ Hence Proved.}$$

Sol 34.

Let $MN = r_3 = MP = MQ, ID = r$

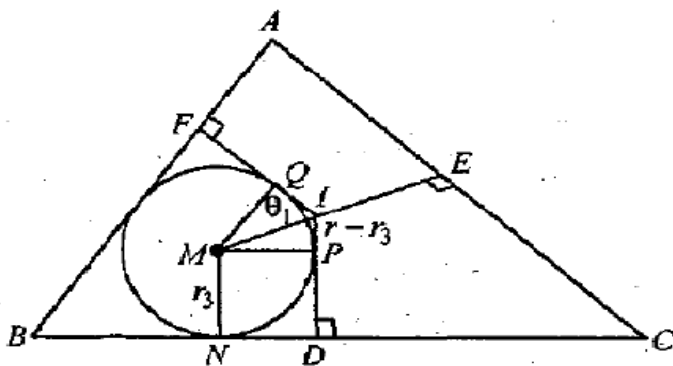
$$\Rightarrow IP = r - r_3$$

Clearly IP and IQ are tangents to circle with centre M .

$\therefore IM$ must be the \angle bisector of $\angle PIQ$

$$\therefore \angle PIM = \angle QIM = \theta_1$$

Also from $\Delta IPM, \tan \theta_1 = r_3/r - r_3 = MP/IP$



Here $ID = r$

Similarly, in other quadrilaterals, we get

$$\tan \theta_2 = r_2/r - r_2 \text{ and } \tan \theta_3 = r_1/r - r_1$$

$$\text{Also } 2\theta_1 + 2\theta_2 + 2\theta_3 = 2\pi \Rightarrow \theta_1 + \theta_2 + \theta_3 = \pi$$

$$\Rightarrow \tan \theta_1 \tan \theta_2 + \tan \theta_1 \tan \theta_3 = \tan \theta_1 \cdot \tan \theta_2 \cdot \tan \theta_3$$

NOTE THIS STEP:

$$= r_1/r - r_1 + r_2/r - r_2 + r/r - r_3 = r_1 r_2 r_3 / (r - r_1) (r - r_2) (r - r_3)$$

Sol 35.

$$\text{We know, } \Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{s/8 (b+c-a)(c+a-b)(a+b-c)}$$

Since sum of two sides is always greater than third side;

$$\therefore b+c-a, c+a-b, a+b-c > 0$$

$$\Rightarrow (s-a)(s-b)(s-c) > 0$$

$$\text{Let } s-a = x, s-b = y, s-c = z$$

$$\text{Now, } x+y = 2s-a-b = c$$

$$\text{Similarly, } y+z = a$$

$$\text{And } z+x = b$$

Since $AM \geq GM$

$$\Rightarrow x+y/2 \geq \sqrt{xy} \Rightarrow 2\sqrt{xy} \leq a \quad y+z/2 \geq \sqrt{yz} \Rightarrow 2\sqrt{yz} \leq b \quad z+x/2 \geq \sqrt{xz} \Rightarrow 2\sqrt{xz} \leq c \therefore 8xyz \leq abc$$

$$\Rightarrow (s-a)(s-b)(s-c) \leq 1/8 abc$$

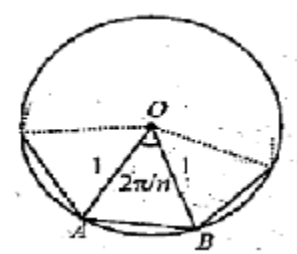
$$\Rightarrow s(s-a)(s-b)(s-c) \leq sabc/8$$

$$\Rightarrow \leq 1/16 (a+b+c) abc \Rightarrow \Delta \leq 1/4 \sqrt{abc} (a+B+c)$$

And equality holds when $x = y = z \Rightarrow a = b = c$

Sol 36.

Let OAB be one triangle out of n on a n sided polygon inscribed in a circle of radius 1.



Then $\angle AOB = 2\pi/n$

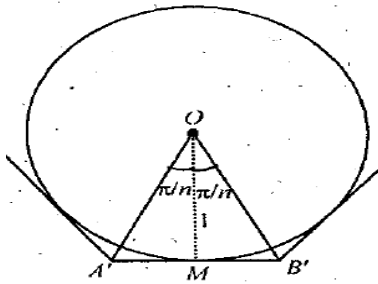
$$OA = OB = r$$

∴ Using Area of isosceles Δ with vertical $\angle \theta$ and equal

$$\text{Sides as } r = \frac{1}{2} r^2 \sin \theta = \frac{1}{2} \sin 2\pi/n$$

$$\therefore I_n = n/2 \sin 2\pi/n \dots \dots \dots (1)$$

Further consider the n sided polygon subscribing on the circle.



A' M' B' is the tangent of the circle at M.

$$\Rightarrow A' M B' \perp OM$$

$\Rightarrow A' M O$ is right angled triangle, right angle at M.

$$A' M = \tan \pi/n$$

$$\text{Area of } \Delta A' M O = \frac{1}{2} \times 1 \times \tan \pi/n$$

$$\therefore \text{Area of } \Delta A' B' O = \tan \pi/n$$

$$\text{So, } O_n = n \tan \pi/n \dots \dots \dots (2)$$

Now, we have to prove

$$I_n = \frac{O_n}{2} \left(1 + \sqrt{1 - \left(\frac{2I_n}{n} \right)^2} \right)$$

$$\text{Or } 2I_n/O_n - 1 = \sqrt{1 - \left(\frac{2I_n}{n} \right)^2}$$

$$\text{LHS} = 2I_n/O_n - 1 = n \sin 2\pi/n / n \tan \pi/n - 1 \text{ (From (1) and (2))}$$

$$= 2 \cos^2 \pi/n - 1 = \cos 2\pi/n$$

$$\text{RHS} = \sqrt{1 - \left(\frac{2I_n}{n} \right)^2} = \sqrt{1 - \sin^2 2\pi/n} \text{ (From (1))}$$

= $\cos (2\pi/n)$ Hence Proved.