

Quadratic Equation and In equations (Inequalities) Solutions

SUBJECTIVE PROBLEMS:

Sol 1.

$$4^x - 3^{x-1/2} = 3^{x+1/2} - (2^2)^{x/2}$$

$$\Rightarrow 4^x - 3^x/\sqrt{3} = 3^x \sqrt{3} - 4^x/2$$

$$\Rightarrow 3/2 \cdot 4^x = 3^x (\sqrt{3} + 1/\sqrt{3}) \Rightarrow 3/2 \cdot 4^x = 3^x \cdot 4/\sqrt{3}$$

$$\Rightarrow (4/3)^{x-3/2} = 1 \Rightarrow x - 3/2 = 0$$

$$\Rightarrow x = 3/2$$

Sol 2.

$$\text{RHS} = (m-1, n+1) + x^{m-n-1} (m-1, n)$$

$$= (1-x^{m-1})(1-x^{m-2}) \dots (1-x^{m-n-1}) / (1-x)(1-x^2) \dots (1-x^{m+1})$$

$$+ x^{m-n-1} [(1-x^{m-1})(1-x^{m-2}) \dots (1-x^{m-n}) / (1-x)(1-x^2) \dots (1-x^n)]$$

$$= (1-x^{m-1})(1-x^{m-2}) \dots (1-x^{m-n}) / (1-x)(1-x^2) \dots (1-x^n)$$

$$[1 - x^{m-n-1} / 1 - x^{n+1} + x^{m-n-1}]$$

$$[1 - x^{m-n-1} + x^{m-n-1} - x^m / 1 - x^{n+1}]$$

$$= (1-x^m)(1-x^{m-1}) \dots (1-x^{m-n}) / (1-x)(1-x^2) \dots (1-x^n)(1-x^{n+1})$$

$$= (m, n+1) = \text{L. H. S.}$$

Sol 3.

$$\sqrt{x+1} = 1 + \sqrt{x-1}$$

Squaring both sides, we get

$$x+1 = 1+x-1+2\sqrt{x-1} \Rightarrow 1 = 2\sqrt{x-1}$$

$$\Rightarrow 1 = 4(x-1)$$

$$\Rightarrow x = 5/4$$

Sol 4.

Given $a > 0$, so we have to consider two cases: $a \neq 1$ and $a = 1$. Also it is clear that $x > 0$

And $x \neq 1$, $ax \neq 1$, $a^2x \neq 1$

Case I: If $a > 0, \neq 1$

Then given equation can be simplified as

$$2/\log_a x + 1/\log_a x + 3/2 + \log_a x = 0$$

Putting $\log_a x = y$, we get

$$2(1 + y)(2 + y) + y(2 + y) + 3y(1 + y) = 0$$

$$\Rightarrow 6y^2 + 11y + 4 = 0 \Rightarrow y = -4/3 \text{ and } -1/2$$

$$\Rightarrow \log_a x = -4/3 \text{ and } \log_a x = -1/2$$

$$\Rightarrow x = a^{-4/3} \text{ and } x = a^{-1/2}$$

Case II: If $a = 1$ then equation becomes

$$2 \log_x 1 + \log_x 1 + 3 \log_x 1 = 6 \log_x 1 = 0$$

Which is true $\forall x > 0, \neq 1$

Hence solution is if $a = 1, x > 0, \neq 1$

If $a > 0, \neq 1; x = a^{-1/2}, a^{-4/3}$

Sol 5.

$$\text{Let } x = \frac{\sqrt{26-15\sqrt{3}}}{5\sqrt{2}-\sqrt{38+5\sqrt{3}}}$$

$$\Rightarrow x^2 = \frac{26-15\sqrt{3}}{50+38+5\sqrt{3}-10\sqrt{76+10\sqrt{3}}}$$

$$\Rightarrow x^2 = \frac{26-15\sqrt{3}}{88+5\sqrt{3}-10\sqrt{71+1+10\sqrt{3}}}$$

$$\Rightarrow x^2 = \frac{26-15\sqrt{3}}{88+5\sqrt{3}-10\sqrt{(5\sqrt{3})^2+(1)^2x5\sqrt{3}x1}}$$

$$\Rightarrow x^2 = \frac{26-15\sqrt{3}}{88+5\sqrt{3}-10\sqrt{(5+\sqrt{3}+1)^2}}$$

$$\Rightarrow x^2 = \frac{26-15\sqrt{3}}{88+5\sqrt{3}-10(5\sqrt{3}+1)} \Rightarrow x^2 = \frac{26-15\sqrt{3}}{88+5\sqrt{3}-50\sqrt{3}-10} = \frac{26-15\sqrt{3}}{78-45\sqrt{3}} = \frac{26-15\sqrt{3}}{3(26-15\sqrt{3})} = \frac{1}{3}, \text{ which is rational number.}$$

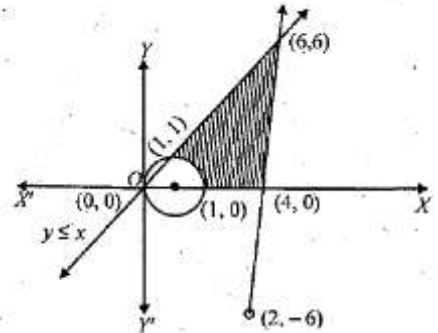
Sol 6.

$$x^2 + y^2 = 2x \geq 0 \Rightarrow x^2 - 2x + 1 + y^2 \geq 1$$

$\Rightarrow (x - 1)^2 + y^2 \geq 1$ which represents the boundary and exterior region of the circle with Centre at (1, 0)

And radius as 1. For $3x - y \leq 12$, the corresponding equation is $3x - y = 12$; any two points on it can be

taken as (4, 0), (2, -6); Also putting (0, 0) in given in equation, $0 \leq 12$ which true.



\therefore given in equation represents that half plane region of line $3x - y = 12$ which contains origin.

For $y \leq x$, the corresponding equation $y = x$ has any two points on it as (0, 0) and (1, 1). Also putting (2, 1)

In the given in equation, we get $1 \leq 2$ which is true, so $y \leq x$ represents that half plane which contains the points (2, 1). $y \geq 0$ represents upper half Cartesian plane.

Combining all we find the solution set as shaded region in the graph.

Sol 7.

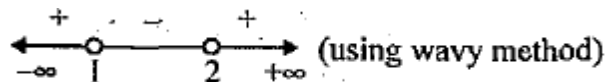
There are two parts of this question

$$(5x - 1) < (x + 1)^2 \text{ and } (x + 1)^2 \text{ and } (x + 1)^2 < (7x - 3) \text{ Taking first part}$$

$$(5x - 1) < (x + 1)^2 \Rightarrow 5x - 1 < x^2 + 2x + 1$$

$$\Rightarrow x^2 - 3x + 2 > 0 \Rightarrow (x - 1)(x - 2) > 0$$

$$\Rightarrow x < 1 \text{ or } x > 2 \dots\dots\dots(1)$$

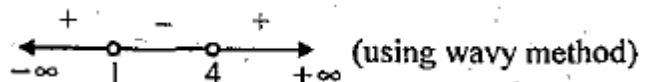


Taking second part

$$(x + 1)^2 < (7x - 3) \Rightarrow x^2 - 5x + 4 < 0$$

$$\Rightarrow (x - 1)(x - 4) < 0$$

$$\Rightarrow 1 < x < 4 \dots\dots\dots(2)$$



Combining (1) and (2) [taking common solution], we get $2 < x < 4$ but x is an integer therefore $x = 3$.

Sol 8.

$\therefore \alpha, \beta$ are the roots of $x^2 + p x + q = 0$

$$\therefore \alpha + \beta = -p, \quad \alpha\beta = q$$

$\therefore \gamma, \delta$ are the roots of $x^2 + r x + s = 0$

$$\therefore \gamma + \delta = -r, \gamma \delta = s$$

Now, $(\alpha - \gamma) (\alpha - \delta) (\beta - \gamma) (\beta - \delta)$

$$= [\alpha^2 - (\gamma + \delta) \alpha + \gamma \delta] [\beta^2 - (\gamma + \delta) \beta + \gamma \delta]$$

$$= \alpha^2 + r\alpha + s [\beta^2 + r\beta + s] \quad [\because \alpha, \beta \text{ are roots of } x^2 + p x + q = 0]$$

$$\therefore \alpha^2 + p\alpha + q = 0 \text{ and } \beta^2 + p\beta + q = 0]$$

$$= [(r - p) \alpha + (s - q)] [(r - p) \beta + (s - q)]$$

$$= (r - p)^2 \alpha \beta + (r - p) (s - q) (\alpha + \beta) + (s - q)^2$$

$$= q(r - p)^2 - p(r - p) (s - q) + (s - q)^2$$

Now if the equations $x^2 + p x + q = 0$ and $x^2 + r x + s = 0$ have a common root say α , then $\alpha^2 + p\alpha +$

$$q = 0 \text{ and } \alpha^2 + r\alpha + s = 0$$

$$\Rightarrow \alpha^2/ps - q r = \alpha/q - s = 1/r - p$$

$$\Rightarrow \alpha^2 = ps - q r/r - p \text{ and } \alpha = q - s/r - p$$

$$\Rightarrow (q - s)^2 = (r - p) (ps - q r) \text{ which is the required condition.}$$

Sol 9.

We know that for sides a, b, c of a Δ

$$(a - b)^2 \geq 0$$

$$\Rightarrow a^2 + b^2 \geq 2ab \quad \dots\dots\dots (1)$$

$$\text{Similarly } b^2 + c^2 \geq 2bc \quad \dots\dots\dots (2)$$

$$c^2 + a^2 \geq 2ca \quad \dots\dots\dots (3)$$

Adding the three in equations, we get

$$2(a^2 + b^2 + c^2) \geq 2(ab + b c + ca)$$

$$\Rightarrow a^2 + b^2 + c^2 \geq ab + b c + ca$$

Adding $2(ab + bc + ca)$ to both sides, we get

$$(a + b + c)^2 \geq 3(ab + bc + ca)$$

$$\text{Or } 3(ab + bc + ca) \leq (a + b + c)^2 \dots\dots\dots (A)$$

Also $c < a + b$ (triangle inequality)

$$\Rightarrow c^2 < ac + bc \dots\dots\dots (4)$$

$$\text{Similarly } b^2 < ab + bc \dots\dots\dots (5)$$

$$a^2 < ab + ca \dots\dots\dots (6)$$

Adding (4), (5) and (6), we get $a^2 + b^2 + c^2 < 2(ab + bc + ca)$

Adding $2(ab + bc + ca)$ to both sides, we get

$$\Rightarrow (a + b + c)^2 < 4(ab + bc + ca) \dots\dots\dots (B)$$

Combining (A) and (B), we get

$$3(ab + bc + ca) \leq (a + b + c)^2 < 4(ab + bc + ca)$$

First two expressions will be equal for $a = b = c$.

Sol 10.

Given that $n^4 < 10^n$ for n for a fixed +ve integer $n \geq 2$.

To prove that $(n + 1)^4, 10^{n+1}$

$$\text{Proof: Since } n^4 < 10^n \Rightarrow 10n^4 < 10^{n+1} \dots\dots\dots (1)$$

So it is sufficient to prove that $(n + 1)^4 < 10n^4$

$$\text{Now } (n + 1/n)^4 = (1 + 1/n)^4 \leq (1 + 1/2)^4 [\because n \geq 2]$$

$$= 81/16 < 10$$

$$\Rightarrow (n + 1)^4 < 10n^4 \dots\dots\dots (2)$$

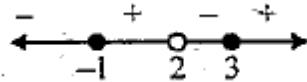
From (1) and (2), $(n + 1)^4 < 10^{n+1}$

Sol 11.

$$Y = \sqrt{\frac{(x+1)(x-3)}{(x-2)}}$$

y will take all real values if $\frac{(x+1)(x-3)}{x-2} \geq 0$

By wavy method:->



$$x \in [-1, 2) \cup [3, \infty)$$

[2 is not included as it makes denominator zero, and hence y an undefined number.]

Sol 12.

The given equations are $3x + my - m = 0$ and $2x - 5y - 20 = 0$ Solving these equations by cross product method, we get

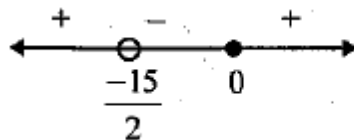
$$x/-20m - 5m = y/-2m + 60 = 1/-15 - 2m \text{ NOTE THIS STEP}$$

$$\Rightarrow x = 25m/2m + 15, y = 2m - 60/2m + 15$$

$$\text{For } x > 0 \Rightarrow 25m/2m + 15 > 0$$

$$\Rightarrow m < -15/2 \text{ or } m > 0 \dots\dots\dots(1)$$

[using wavy method]:->

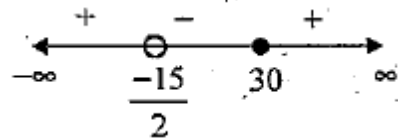


$$\text{For } y > 0 \Rightarrow 2(m - 30)/2m + 15 = 0$$

$$\Rightarrow m < -15/2 \text{ or } m >$$

$$30 \dots\dots\dots(2)$$

[using wavy method]:->



Combining (1) and (2), we get the common values of m as follows:

$$m < 15/2 \text{ or } m > 30 \therefore m \in (-\infty, -15/2) \cup (30, \infty)$$

Sol 13.

The given system is

$$x + 2y + z = 1 \dots\dots\dots(1)$$

$$2x - 3y - \omega \geq 0$$

Multiplying eqn. (1) by (2) and subtracting from (2), we get

$$7y + 2z + \omega = 0 \Rightarrow \omega = -(7y + 2z)$$

Now if $y, z > 0, \omega < 0$ (not possible)

If $y = 0, z = 0$ then $x = 1$ and $\omega = 0$.

∴ The only solution is $x = 1, y = 0, z = 0, \omega = 0$.

Sol 14.

$$e^{\sin x} - e^{-\sin x} - 4 = 0$$

Let $e^{\sin x} = y$ then $e^{-\sin x} = 1/y$

∴ Equation becomes, $y - 1/y - 4 = 0$

$$\Rightarrow y^2 - 4y - 1 = 0$$

$$\Rightarrow y = 2 + \sqrt{5}, 2 - \sqrt{5}$$

But y is real +ve number,

$$\therefore y \neq 2 - \sqrt{5} \Rightarrow y = 2 + \sqrt{5}$$

$$\Rightarrow e^{\sin x} = 2 + \sqrt{5} \Rightarrow \sin x = \log_e (2 + \sqrt{5})$$

But $2 + \sqrt{5} > e \Rightarrow \log_e (2 + \sqrt{5}) > \log_e e$

$\Rightarrow \log_e (2 + \sqrt{5}) > 1$ Hence, $\sin x > 1$

Which is not possible ∴ Given equation has no real solution.

Sol 15.

For any square can be at most 4, neighbours squares.

Let for a square having largest number d , p, q, r, s be written then

According to the question,

$$p + q + r + s = 4d$$

$$\Rightarrow (d - p) + (d - q) + (d - r) + (d - s) = 0$$

Sum of four +ve numbers can be zero only if these are zero individually

$$\therefore d - p = 0 = d - q = d - r = d - s$$

$$\Rightarrow p = q = r = s = d$$

\Rightarrow all the numbers written are same. Hence Proved.

		q			
	p	d	r		
		s			

Sol 16.

Let α, β be the roots of eq. $ax^2 + bx + c = 0$

According to the question,

$$B = \alpha^n$$

$$\text{Also } \alpha + \beta = -b/a; \alpha\beta = c/a$$

$$\alpha\beta = c/a \Rightarrow \alpha \cdot \alpha^n = c/a \Rightarrow \alpha = (c/a)^{1/n+1}$$

$$\text{then } \alpha + \beta = -b/a \Rightarrow \alpha + \alpha^n = -b/a$$

$$\text{Or } (c/a)^{1/n+1} + (c/a)^{n/n+1} = -b/a$$

$$\Rightarrow a \cdot (c/a)^{1/n+1} + a \cdot (c/a)^{n/n+1} + b = 0$$

$$\Rightarrow a^{n/n+1} c^{1/n+1} + a^{1/n+1} c^{n/n+1} + b = 0$$

$$\Rightarrow (a^n c)^{1/n+1} + (can)^{1/n+1} + b = 0 \text{ Hence proved.}$$

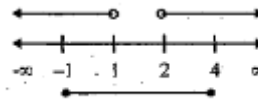
Sol 17.

$$x^2 - 3x + 2 > 0, x^2 - 3x - 4 \leq 0$$

$$\Rightarrow (x - 1)(x - 2) > 0 \text{ and } (x - 4)(x + 1) < 0$$

$$\Rightarrow x \in (-\infty, 1) \cup (2, \infty) \text{ and } x \in [-1, 4]$$

$$\therefore \text{Common solution is } [-1, 1) \cup (2, 4]$$



Sol 18.

The given equation is

$$(5 + 2\sqrt{6})^{x^2-3} + (5 - 2\sqrt{6})^{x^2-3} = 10 \dots\dots\dots (1)$$

$$\text{Let } (5 + 2\sqrt{6})^{x^2-3} = y \dots\dots\dots (2)$$

$$\text{Then } (5 - 2\sqrt{6})^{x^2-3} = \left(\frac{(5-2\sqrt{6})(5+2\sqrt{6})}{5+2\sqrt{6}}\right)^{x^2-3}$$

$$= \left(\frac{25-24}{5+2\sqrt{6}}\right)^{x^2-3} = \left(\frac{1}{5+2\sqrt{6}}\right)^{x^2-3} = 1/y \text{ (Using (2))}$$

\therefore The given equation (1) becomes $y + 1/y = 10$

$$\Rightarrow y^2 - 10y + 1 = 0$$

$$\Rightarrow y = 10 \pm \sqrt{100-4}/2 = 10 \pm 4\sqrt{6}/2$$

$$\Rightarrow y = 5 + 2\sqrt{6} \text{ or } 5 - 2\sqrt{6}$$

Consider $y = 5 + 2\sqrt{6}$

$$\Rightarrow (5 + 2\sqrt{6})^{x^2-3} = (5 + 2\sqrt{6})$$

$$\Rightarrow x^2 - 3 = 1 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$$

Again consider

$$y = 5 - 2\sqrt{6} = 1/5 + 2\sqrt{6} = (5 + 2\sqrt{6})^{-1}$$

$$\Rightarrow (5 + 2\sqrt{6})^{x^2-3} = (5 + 2\sqrt{6})^{-1} \Rightarrow x^2 - 3 = -1$$

$$\Rightarrow x^2 = 2$$

$$\Rightarrow x = \pm \sqrt{2} \text{ Hence the solution are } 2, -2, \sqrt{2}, -\sqrt{2}.$$

Sol 19.

The given equation is,

$$x^2 - 2a|x - a| - 3a^2 = 0$$

Here two cases are possible

Case I: $x - a > 0$ then $|x - a| = x - a$

\therefore Eq. Becomes

$$x^2 - 2a(x - a) - 3a^2 = 0$$

$$\text{Or } x^2 - 2ax - a^2 = 0 \Rightarrow x = 2a \pm \sqrt{4a^2 + 4a^2}/2$$

$$\Rightarrow x = a \pm a\sqrt{2}$$

Case II: $x - a < 0$ then $|x - a| = -(x - a)$

\therefore Eq. becomes

$$x^2 + 2a(x - a) - 3a^2 = 0$$

$$\text{Or } x^2 + 2ax - 5a^2 = 0$$

$$\Rightarrow x = -2a \pm \sqrt{4a^2 + 20a^2}/2 \Rightarrow x = -2a \pm 2a\sqrt{6}/2$$

$$x = -a \pm a\sqrt{6} \text{ Thus the solution set is } \{a \pm a\sqrt{2}, -a \pm a\sqrt{6}\}$$

Sol 20.

We are given $2x/2x^2 + 5x + 2 > 1/x + 1$

$$\Rightarrow 2x/2x^2 + 5x + 2 - 1/x + 1 > 0$$

$$\Rightarrow 2x^2 + 2x - 2x^2 - 5x - 2 / (2x^2 + 5x + 2) (x + 1) > 0$$

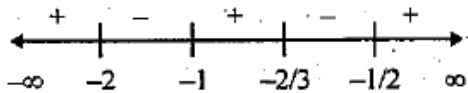
$$\Rightarrow -3x - 2 / (2x + 1) (x + 1) (x + 2) > 0$$

$$\Rightarrow (3x + 2) / (x + 1) (x + 2) (2x + 1) < 0$$

$$\Rightarrow (3x + 2) (x + 1) (x + 2) (2x + 1) / (x + 1)^2 (x + 2)^2 (2x + 1)^2 < 0$$

$$\Rightarrow (3x + 2) (x + 1) (x + 2) (2x + 1) < 0 \dots\dots\dots (1)$$

NOTE THIS STEP: Critical pts. Are $x = -2/3, -1, -2, -1/2$ on number line



Clearly Inequality (1) holds for,

$$x \in (-2, -1) \cup (-2/3, -1/2) \text{ [as } x \neq -2, -1, -2/3, -1/2]$$

Sol 21.

We are given that α_1, α_2 are the roots of

$$ax^2 + bx + c = 0$$

$$\therefore \alpha_1 + \alpha_2 = -b/a; \alpha_1 \alpha_2 = c/a \dots\dots\dots (1)$$

And β_1, β_2 are the roots of $px^2 + qx + r = 0$

$$\therefore \beta_1 + \beta_2 = -q/p; \beta_1 \beta_2 = r/p \dots\dots\dots (2)$$

The system of equations, $\alpha_1 y + \alpha_2 z = 0$

And $\beta_1 y + \beta_2 z = 0$ has a non trivial solution.

$$\therefore \text{we must have } |\alpha_1 \beta_1 \alpha_2 \beta_2| = 0$$

NOTE THIS STEP:

$$\Rightarrow \alpha_1 \beta_2 - \alpha_2 \beta_1 = 0 \Rightarrow \alpha_1 / \alpha_2 = \beta_1 / \beta_2$$

By componendo and dividend, we get

$$\alpha_1 + \alpha_2 / \alpha_1 - \alpha_2 = \beta_1 + \beta_2 / \beta_1 - \beta_2$$

$$\Rightarrow (\alpha_1 + \alpha_2) (\beta_1 - \beta_2) = (\alpha_1 - \alpha_2) (\beta_1 + \beta_2)$$

$$\Rightarrow (\alpha_1 + \alpha_2)^2 [(\beta_1 - \beta_2)^2 - 4\beta_1 \beta_2]$$

$$= [(\alpha_1 + \alpha_2)^2 - 4\alpha_1\alpha_2] (\beta_1 + \beta_2)^2$$

Using equations (1) and (2) we get

$$b^2/a^2 [q^2/p^2 - 4r/p] = q^2/p^2 [b^2/a^2 - 4c/a]$$

$$\Rightarrow b^2 q^2 / a^2 p^2 - 4b^2 r / a^2 p = q^2 b^2 / q^2 a^2 - 4cq^2 / ap^2 \Rightarrow -4b^2 r / a^2 p = -4cq^2 / ap^2 \Rightarrow b^2 r / a = sq^2 / p \Rightarrow b^2 / q^2 = ac / p r \text{ Hence Proved}$$

Sol 22.

The Given equation is,

$$|x^2 + 4x + 3| + 2x + 5 = 0$$

Now there can be two cases.

Case I: $x^2 + 4x + 3 \geq 0 \Rightarrow (x + 1) (x + 3) \geq 0$

$$\Rightarrow x \in (-\infty, -3] \cup [-1, \infty) \dots\dots\dots (i)$$

Then given equation becomes,

$$\Rightarrow x^2 + 6x + 8 = 0$$

$$\Rightarrow (x + 4) (x + 2) = 0 \Rightarrow x = -4, -2$$

But $x = -2$ does not satisfy (i), hence rejected

$\therefore x = -4$ is the sol.

Case II: $x^2 + 4x + 3 < 0$

$$\Rightarrow (x + 1) (x + 3) < 0$$

$$\Rightarrow x \in (-3, -1) \dots\dots\dots (ii)$$

Then given equation becomes,

$$-(x^2 + 4x + 3) + 2x + 5 = 0$$

$$\Rightarrow -x^2 - 2x + 2 = 0 \Rightarrow x^2 + 2x - 2 = 0$$

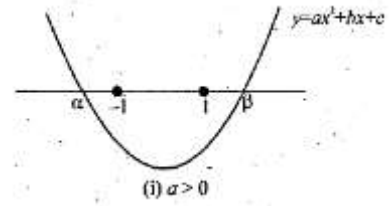
$$\Rightarrow x = -2 \pm \sqrt{4 + 8} / 2 \Rightarrow x = -1 + \sqrt{3}, -1 - \sqrt{3}$$

Out of which $x = -1 - \sqrt{3}$ is sol. Combining the two cases we get the solutions of given equation as $x = -4, -1 - \sqrt{3}$.

Sol 23.

Given that for $a, b, c \in \mathbb{R}$, $ax^2 + bx + c = 0$ has two real roots α and β , where $\alpha < -1$ and $\beta > 1$. There may be two cases depending upon value of a , as shown below.

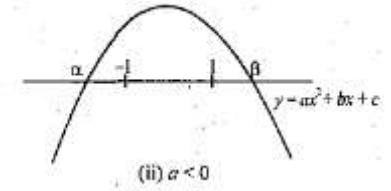
In each of cases (i) and (ii) of $(-1) < 0$ and of $(1) < 0$



$$\Rightarrow a(a - b + c) < 0 \text{ and } a(a + b + c) < 0$$

Dividing by $a^2 (>0)$, we get

$$1 - b/a + c/a < 0 \dots\dots\dots (1)$$



$$\text{And } 1 + b/a + c/a < 0 \dots\dots\dots (2)$$

Combining (1) and (2) we get

$$1 + \left| \frac{b}{a} \right| + \frac{c}{a} < 0 \text{ or } 1 + \frac{c}{a} + \left| \frac{b}{a} \right| < 0 \text{ Hence Proved.}$$

Sol 24.

The given equation is,

$$2|y| - |2^{y-1} - 1| = 2^{y-1} + 1$$

On the basis of absolute values involved here ($|y|$ and $|2^{y-1} - 1|$), there are two critical pts 0 and 1.

So we shall consider three cases, when y lies in three different intervals namely $(-\infty, 0)$, $[0, 1]$,

$(1, \infty)$

Case I: $y \in (-\infty, 0)$ then

$$|y| = -y \text{ and } |2^{y-1} - 1| = 1 - 2^{y-1}$$

\therefore The given equation becomes

$$2^{-y} - 1 + 2^{y-1} = 2^{y-1} + 1$$

$$\Rightarrow 2^{-y} = 2 \Rightarrow y = -1 \in (-\infty, 0)$$

Case II: $y \in [0, 1]$

$$\text{If } y = 0 \text{ we get } 1 - |1/2 - 1| = 1/2 + 1$$

$$1 - 1/2 = 1/2 + 1 \quad (\text{not satisfied})$$

$\therefore y = 0$ is not a solⁿ

If $y = 1$ we get $2 - |2^0 - 1| = 2^0 + 1$

$\Rightarrow 2 = 2$ (satisfied)

$\therefore y = 1$ is a solⁿ

If $y \in (0, 1)$ then $|y| = y$ and $|2^{y-1} - 1| = 1 - 2^{y-1}$

\therefore the eq. ⁿ becomes

$$2^y - 1 + 2^{y-1} + 1 \Rightarrow 2^y = 2$$

$\Rightarrow y = 1 \notin (0, 1)$

$\therefore y = 1$ is the only solⁿ in this case.

Case III: $y \in (1, \infty)$

Then $|y| = 1$, $|2^{y-1} - 1| = 2^{y-1} - 1$

The given eq. ⁿ becomes, $2^y - 2^{y-1} + 1 = 2^{y-1} + 1$

$$\Rightarrow 2^y - 2^y = 0$$

Which is satisfied for all real values of y but $y \in (1, \infty)$

$\therefore (1, \infty)$ is the solⁿ in this case.

Combining all the cases, we get the solⁿ as $y \in \{1\} \cup [1, \infty]$

Sol 25.

$$a^2 = p^2 + s^2, b^2 = (1 - p)^2 + q^2$$

$$c^2 = (1 - q)^2 + (1 - r)^2, a^2 = r^2 + (1 - s)^2$$

$$\therefore a^2 + b^2 + c^2 + d^2 = \{p^2 + (1 - p)^2\} + \{q^2 - (1 - q)^2\} + \{r^2\} + (1 - r)^2 + \{s^2 + (1 - s)^2\}$$

Where p, q, r, s all vary in the interval $[0, 1]$.

Now consider the function

$$y^2 = x^2 + (1 - x)^2, 0 \leq x \leq 1,$$

$$2y \frac{dy}{dx} = 2x - 2(1 - x) = 0$$

$$\Rightarrow x = 1/2 \text{ which } \frac{d^2 y}{dx^2} = 4 \text{ i.e. +ve}$$

Hence y is minimum at $x = 1/2$ and it's minimum

Value is $1/4 + 1/4 = 1/2$

Clearly value is maximum at the end pts which is 1.

\therefore Minimum value of $a^2 + b^2 + c^2 + d^2 = 1/2 + 1/2 + 1/2 + 1/2 = 2$

And maximum value is $1 + 1 + 1 + 1 = 4$. Hence proved.

ALTERNATE SOLUTION:

$$x^2 + y^2 \leq (x + y)^2 \leq 1 \text{ if } x + y = 1$$

Here $x = p, y = 1 - p \therefore x + y = 1$

$$\therefore a^2 + b^2 + c^2 + d^2 \leq 1 + 1 + 1 + 1 = 4$$

$$\text{Again } x^2 + y^2 = (x + y)^2 - 2xy = 1 - 2xy$$

NOTE THIS STEP: \therefore Minimum of $(x^2 + y^2) = 1 - 2(\text{maximum of } xy)$.

Now we know that product of two quantities xy is maximum when the quantities are equal provided their sum is constant.

Here $x + y = p + 1 - p = 1 = \text{constant}$.

$$\therefore xy \text{ is maximum when } x/1 = y/1 = x + y/2 = 1/2$$

$$\therefore x = 1/2, y = 1/2$$

$$\text{Minimum of } x^2 + y^2 = 1 - 2 \cdot 1/2 \cdot 1/2 = 1 - 1/2 = 1/2$$

\therefore Minimum value of

$$a^2 + b^2 + c^2 + d^2 = 1/2 + 1/2 + 1/2 + 1/2 = 2$$

$$\therefore 2 \leq a^2 + b^2 + c^2 + d^2 \leq 4.$$

Sol 26.

We know that,

$$(\alpha - \beta)^2 = [(\alpha + \delta) - (\beta + \delta)]^2$$

$$\Rightarrow (\alpha + \beta)^2 - 4\alpha\beta = (\alpha + \delta + \beta + \delta)^2 - 4(\alpha + \delta)(\beta + \delta)$$

$$\Rightarrow b^2/a^2 - 4c/a = B^2/A^2 - 4C/A \Rightarrow 4ac - b^2/a^2 = 4AC - B^2/A^2$$

[Hence $\alpha + \beta = -b/a, \alpha\beta = c/a$

$(\alpha + \delta)(\beta + \delta) = -B/A$ and $(\alpha + \delta)(\beta + \delta) = C/A$] Hence proved.

Sol 27.

$$\alpha + \beta = -b/a, \alpha\beta = c/a$$

Roots of the equation $a^3 x^3 + abcx + c^3 = 0$ are

$$\begin{aligned}x &= -abc \pm \sqrt{(abc)^2 - 4a^3 c^3} / 2a^3 \\&= (-b/a)(c/a) \pm \sqrt{(b/a)^2 (c/a)^2 - 4(c/a)^3} / 2 \\&= (\alpha + \beta)(\alpha\beta) \pm \sqrt{(\alpha + \beta)^2 (\alpha\beta)^2 - 4(\alpha\beta)^3} / 2 \\&= (\alpha\beta) ((\alpha + \beta) \pm \sqrt{(\alpha - \beta)^2}) / 2 \\&= (\alpha\beta) ((\alpha + \beta) \pm (\alpha - \beta)) / 2 = \alpha^2 \beta, \alpha \beta^2\end{aligned}$$

Let γ and δ be the required roots. Then

$$\gamma = \alpha^2 \beta \text{ and } \delta = \alpha \beta^2.$$

ALTERNATE SOLUTION:

$ax^2 + bx + c = 0$ has roots α and β . (given)

$$\Rightarrow \alpha + \beta = -b/a \text{ and } \alpha\beta = c/a$$

Now, $a^3 x^3 + abcx + c^3 = 0$

Divides the equation by c^2 , we get

$$a^3/c^2 x^3 + abcx/c^2 + c^3/c^2 = 0, a(ax/c)^2 + b(ax/c) + c = 0$$

$$\Rightarrow ax/c = \alpha, \beta \text{ are the roots}$$

$$\Rightarrow x = c/a \alpha, c/a \beta \text{ are the roots}$$

$$\Rightarrow x = \alpha \beta \alpha, \alpha \beta \beta \text{ are the roots}$$

$$\Rightarrow x = \alpha^2 \beta, \alpha \beta^2 \text{ are the roots}$$

ALTERNATE SOLUTION:

Divide the equation by a^3 , we get

$$x^3 + b/a \cdot c/a \cdot x + (c/a)^3 = 0$$

$$\Rightarrow x^3 - (\alpha + \beta)(\alpha\beta)x + (\alpha\beta)^3 = 0 \Rightarrow x^3 - \alpha^2 \beta x - \alpha \beta^2 x + (\alpha\beta)^3 = 0$$

$$\Rightarrow x(x - \alpha^2 \beta) - \alpha \beta^2 (x - \alpha^2 \beta) = 0 \Rightarrow (x - \alpha^2 \beta)(x - \alpha \beta^2) = 0$$

$$\Rightarrow x = \alpha^2 \beta, \alpha \beta^2 \text{ which is the required answer.}$$

Sol 28.

The given equation is,

$$x^2 + (a - b)x + (1 - a - b) = 0$$

$a, b \in \mathbb{R}$ For the eq. ⁿ to have unequal real roots $\forall b \quad D > 0$

$$\Rightarrow (a - b)^2 - 4(1 - a - b) > 0$$

$$\Rightarrow a^2 + b^2 - 2ab - 4 + 4a + 4b > 0$$

$$\Rightarrow b^2 + b(4 - 2a) + a^2 + 4a - 4 > 0$$

Which is a quadratic expression in b, and it will be true $\forall b \in \mathbb{R}$ if discriminant of above of above eq. ⁿ less than zero.

$$\text{i.e., } (4 - 2a)^2 - 4(a^2 + 4a - 4) < 0$$

$$\Rightarrow (2 - a)^2 - (a^2 + 4a - 4) < 0$$

$$\Rightarrow 4 - 4a + a^2 - a^2 - 4a + 4 < 0$$

$$\Rightarrow -8a + 8 < 0 \Rightarrow a > 1$$

Sol 29.

Given that a, b, c are positive real numbers. To prove that $(a + 1)^7 (b + 1)^7 (c + 1)^7 > 7^7 a^4 b^4 c^4$

$$\text{Consider L. H. S.} = (1 + a)^7 (1 + b)^7 (1 + c)^7$$

$$= [(1 + a)(1 + b)(1 + c)]^7$$

$$[1 + a + b + c + ab + bc + ca + abc]^7 > [a + b + c + ab + bc + ca + abc]^7 \dots\dots\dots(1)$$

Now we know that AM \geq GM using it for + ve no's a, b, c, ab, bc, ca and abc, we get

$$\Rightarrow (a + b + c + ab + bc + ca + abc)^7 \geq 7^7 (a^4 b^4 c^4)$$

From (1) and (2), we get

$$[(1 + a)(1 + b)(1 + c)]^7 > 7^7 a^4 b^4 c^4$$

Hence proved.

Sol 30.

Roots of $x^2 - 10cx - 11d = 0$ are a and b

$$\Rightarrow a + b = 10c \text{ and } ab = -11d$$

Similarly c and d are the roots of $x^2 - 10ax - 11b = 0$

$$\Rightarrow c + d = 10a \text{ and } cd = -11b$$

$$\Rightarrow a + b + c + d = 10(a + c) \text{ and } abcd = 121 \text{ bd}$$

$$\Rightarrow b + d = 9(a + c) \text{ and } ac = 121$$

Also we have $a^2 - 10ac - 11d = 0$ and $c^2 - 10ac - 11b = 0$

$$\Rightarrow a^2 + c^2 - 20ac - 11(b + d) = 0$$

$$\Rightarrow (a + c)^2 - 22 \times 121 - 99(a + c) = 0$$

$$\Rightarrow a + c = 121 \text{ or } -22$$

For $a + c = -22$, we get $a = c$

\therefore rejecting this value we have $a + c = 121$

$$\therefore a + b + c + d = 10(a + c)$$