Sequences and Series-Solutions

SUBJECTIVE PROBLEMS :

Sol. 1.

Let the two numbers be a and b, then

$2ab/a + b = 4$ ....(1); $a + b/2 = A$; $\sqrt{ab} = G$

Also $2A + G^2 = 27 \Rightarrow a + b + ab = 27$ ………… (2)

Putting $ab = 27 - (a + b)$ in eqn. (1). We get

$54 - 2(a + b)/a + b = 4 \Rightarrow a + b = 9$ then $ab = 27 - 9 = 18$

Solving the two we gat $a = 6, b = 3$ or $a = 3, b = 6$, which are the required numbers.

Sol. 2.

Let there be in n sides in the polygon.

Then by geometry, sum of all n interior angles of polygon = $(n - 2) * 180°$

Also the angles are in A. P. with the smallest angle = $120°$, common difference = $5°$

∴ Sum of all interior angles of polygon

$= n/2[2 * 120 + (n - 1) * 5$

Thus we should have

$n/2 [2 * 120 + (n - 1) * 5] = (n - 2) * 180$

$\Rightarrow n/2 [5n + 235] = (n - 2) * 180$

$\Rightarrow 5n^2 + 235n = 360n - 720$

$\Rightarrow 5n^2 - 125n + 720 = 0 \Rightarrow n^2 - 25n + 144 = 0$

$\Rightarrow (n - 16) (n - 9) = 0 \Rightarrow n = 16, 9$

Also if $n = 16$ then 16^{th} angle $= 120 + 15 * 5 = 195° > 180°$

∴ not possible. Hence $n = 9$.

Sol. 3.

a_1, a_2 . . . . a_n are in A. P. $\forall a_i > 0$

$\therefore a_1 - a_2 = a_2 - a_3 =$ . . . . $= a_{n-1} - a_n = d$ (a constant) . . . . (1)

Now we have to prove

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \ldots\ldots + \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}} = \frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$$

L. H. S. $= \frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \ldots\ldots + \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}$

$= \frac{\sqrt{a_1}-\sqrt{a_2}}{a_1-a_2} + \frac{\sqrt{a_2}-\sqrt{a_3}}{a_2-a_3} + \ldots\ldots + \frac{\sqrt{a_{n-1}}-\sqrt{a_n}}{a_{n-1}-a_n}$

$= \frac{1}{d}[\sqrt{a_1} - \sqrt{a_2} + \sqrt{a_2} - \sqrt{a_3} + \ldots\ldots + \sqrt{a_{n-1}} - \sqrt{a_n}]$

(Using equation (1))

$= \frac{1}{d}[\sqrt{a_1} - \sqrt{a_n}]$

$= \frac{a_1-a_n}{d(\sqrt{a_1}+\sqrt{a_n})} = \frac{(n-1)d}{d\sqrt{a_1}+\sqrt{a_n}} = \frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$

= R H S. Hence Proved.

Sol. 4.

If possible let for a G. P.

$T_p = 27 = AR^{p-1}$ .... (1)

$T_q = 8 = AR^{q-1}$ .... (2)

$T_r = 12 = AR^{r-1}$ .... (3)

From (1) and (2)

$R^{p-q} = 27/8 \Rightarrow R^{p-q} = (3/2)^3$ .... (4)

From (2) and (3) ;

$R^{q-r} = 8/12 \Rightarrow R^{q-r} = (3/2)$ -1 .... (5)

From (4) and (5)

$R = 3/2$; $p-q = 3$; $q - r = -1$

$p - 2q + r = 4; p, q, r \in N$ ………… (6)

As there can be infinite natural numbers for p, q, and r to satisfy equation (6)

∴ There can be infinite G. P' s.

Sol. 5.

$2 < a, b, c, < 18 \quad a + b + c = 25$ ……. (1)

2, a, b are AP $\Rightarrow 2a = b + 2$

$\Rightarrow 2a - b = 2$ …….. (2)

b, c, 18 are in GP $\Rightarrow c^2 = 18b$ …… (3)

From (2) $\Rightarrow a = b + 2/2$

(1) $\Rightarrow b + 2/2 + b + c = 25 \Rightarrow 3b = 48 - 2c$

(3) $\Rightarrow c^2 = 6(48 - 2c) \Rightarrow c^2 + 12c - 288 = 0$

$\Rightarrow c = 12, -24$ (rejected)

$\Rightarrow a = 5, b = 8, c = 12$

Sol. 6.

Given that a, b, c > 0

We know for +ve numbers A. M. ≥ G. M.

∴ For +ve numbers a, b, c we get

$a + b + c/3 \geq \sqrt[3]{abc}$ . . . . (1)

Also for +ve numbers 1/a, 1/b, 1/c, we get

$\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3} \geq \sqrt[3]{\frac{1}{abc}}$ . . . . (2)

Multiplying in eqs (1) and (2) we get

$\left(\frac{a+b+c}{3}\right)\left(\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3}\right) \geq \sqrt[3]{abc} \times \frac{1}{\sqrt[3]{abc}}$

$\Rightarrow (a + b + c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$ Proved.

Sol. 7.

Given that $n = p_1^{\alpha 1} . p_2^{\alpha 2} . p_3^{\alpha 3} \ldots\ldots\ldots\ldots\ldots\ldots\ldots p^{\alpha k}{}_k$ ……………………. (1)

Where $n \in N$ and $p_1, p_2, p_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots p_k$ are distinct

Prime numbers.

Taking log on both sides of eq. (1), we get

$\log n = \alpha_1 \log p_1 + \alpha_2 \log p_2 + \ldots\ldots\ldots\ldots\ldots\ldots + \alpha_k \log p_k$ ……………………. (2)

Since every prime number is such that

$p_i \geq 2$

$\therefore \log_e p_i \geq \log_e 2$ ……………………. (3)

$\forall$ I = 1 (1) k

Using (2) and (3) we get

$\log n \geq \alpha_1 \log 2 + \alpha_2 \log 2 + \alpha_3 \log 2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + \alpha_k \log 2$

$\Rightarrow \log n \geq (\alpha_1 + \alpha_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + \alpha_k) \log 2$

$\Rightarrow \log n \geq k \log 2$ Proved

Sol. 8.

The given series is

$\sum_{r=0}^{n} (-1)^r\, {}^nC_r \left[\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{2r}} + \frac{15^r}{2^{4r}} + \ldots\ldots\ldots\ldots\ldots\ldots \text{up to m terms}\right]$

$\sum_{r=0}^{n} (-1)^r\, {}^nC_r \left(\frac{1}{2}\right)^r + \left(\frac{3}{4}\right)^r + \left(\frac{7}{8}\right)^r + \left(\frac{15}{16}\right)^r + \ldots\ldots\ldots\ldots\ldots\ldots$ to m terms]

Now, $\sum_{r=0}^{n} (-1)^r\, {}^nC_r \left(\frac{1}{2}\right)^r = 1 - {}^nC_1 . 1/2 + {}^nC_2 . ½^2 - {}^nC_3 . ½^3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$= \left(1 - \frac{1}{2}\right)^n = \frac{1}{2^n}$

Similarly, $\sum_{r=0}^{n} (-1)^r\, {}^nC_r \left(\frac{3}{4}\right)^r = \left(1 - \frac{3}{4}\right)^n = \frac{1}{4^n}$ etc.

Hence the given series is, $= \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{8^n} + \frac{1}{16^n} + \ldots\ldots\ldots\ldots\ldots$ to m terms

$= \frac{\frac{1}{2^n}\left(1 - \left(\frac{1}{2^n}\right)^m\right)}{1 - \frac{1}{2^n}}$ [Summing the G. P.] $= \frac{2^{mn} - 1}{2^{mn}(2^n - 1)}$

Sol. 9.

Let the three distinct real numbers be $\alpha/r, \alpha, \alpha r$.

Since sum of squares of three numbers be S^2

$\therefore \alpha^2/r^2 + \alpha^2 + \alpha^2 r^2 = S^2$

Or $\alpha^2(1 + r^2 + r^4)/r^2 = S^2$ .... (1)

Sum of numbers is aS

$\therefore \alpha (1 + r + r^2)/r = aS$ .... (2)

Dividing eq. (1) by the square (2), we get

$\alpha^2 (1 + r^2 + r^4)/r^2 * r^2/ \alpha^2 (1 + r + r^2)^2 = S^2/a^2S^2$

$(1 + 2r^2 + r^4) - r^2 / (1 + r + r^2)^2 = 1/a^2, (1 + r + r^2) (1 - r + r^2) /(1 + r + r^2)^2 = 1/a^2$

$1 - r + r^2/1 + r + r^2 = 1/a^2 \Rightarrow a^2r^2 - a^2r + a^2 = 1 + r + r^2$

$\Rightarrow (a^2 - 1) r^2 - (a^2 + 1) r + (a^2 - 1) = 0$

$\Rightarrow r^2 + (1 + a^2/1 - a^2)r + 1 = 0$ ...... (3)

For real valuew of r, $D \geq 0$

$\Rightarrow (1 + a^2/1 - a^2)^2 - 4 \geq 0$

$\Rightarrow 1 + 2a^2 + a^2 - 4 + 8a^2 - 4^2 \geq 0$

$\Rightarrow 3a^4 - 10a^2 + 3 \leq 0 \Rightarrow (3a^2 - 1) (a^2 - 3) \leq 0$

$\Rightarrow (a^2 - 1/3) (a^2 - 3) \leq 0$

Clearly the above inequality holds for

$1/3 \leq a^2 \leq 3$

But from eq. (3), $a \neq 1 \therefore a^2 \in (1/3, 1) \cup (1 , 3)$.

Sol. 10.

The given equation is

$\log_{(2x + 3)} (6x^2 + 23x + 21)$

$= 4 - \log_{3x+7} (4x^2 + 12x + 9)$

$\Rightarrow \log_{(2x+3)} (6x^2 + 23x\ 21)$

$+ \log_{(3x+7)} (4x^2 + 12x + 9) = 4$

$\Rightarrow \log_{(2x+3)} (2x + 3) (3x + 7) + \log_{(3x+7)} (2x + 3)^2 x = 4$

$\Rightarrow 1 + \log_{(2x+3)} (3x + 7) + 2 \log_{(3x+7)} (2x + 3) = 4$

[Using $\log ab = a + \log b$ and $\log a^n = n \log a$]

NOTE THIS STEP

$\Rightarrow \log_{(2x+3)} (3x + 7) + 2/ \log_{(2x+3)} (3x + 7) = 3$ [Using $\log_a b = 1/\log_b a$]

Let $\log \log_{(2x+3)} (3x + 7) = y$

$\Rightarrow y + 2/y = 3 \Rightarrow y^2 - 3y + 2 = 0$

$\Rightarrow (y - 1) (y - 2) = 0 \Rightarrow y = 1, 2$

Substituting the values of y in (1), we get

$\Rightarrow \log_{(2x+3)} (3x + 7) = 1$ and $\log_{(2x+3)} (3x + 7) = 2$

$\Rightarrow 3x + 7 = 2x + 3$ and $3x + 7 = (2x + 3)^2$

$\Rightarrow x = -4$ and $4x^2 + 9x + 2 = 0$

$\Rightarrow x = -4$ and $(x + 2) (4x + 1) = 0$

$\Rightarrow x = -4$ and $x = 02, x = -1/4$

As $\log_a x$ is defined for $x > 0$ and $a > 0$ $(a \neq 1)$, the possible value of x should satisfy all of the following inequalities :

$\Rightarrow 2x + 3 > 0$ and $3x + 7 > 0$

Also $2x + 3 \neq 1$ and $3x + 7 \neq 1$

Out of $x = -4$, $x = -2$ and $x = -1/4$ only $x = -1/4$

Satisfies the above inequalities.

So only solutions is $x = -1/4$.

Sol. 11.

Given that $\log_3 2, \log_3(2^x - 5), \log_3 (2^x - 7/2)$ are in A. P.

$\Rightarrow 2 \log_3 (2^x - 5) = \log_3 2 + \log_3 (2^x - 7/2)$

$\Rightarrow (2^x - 5)^2 = 2 (2^x - 7/2)$

$\Rightarrow (2x)^2 - 10.2^x + 25 - 2.2^x + 7 = 0$

$\Rightarrow (2^x)^2 - 10.2^x + 25 - 2.2^x + 7 = 0$

$\Rightarrow (2^x)^2 - 12.2^x + 32 = 0$

Let $2^x = y$, then we get,

$Y^2 - 12y + 32 = 0 \Rightarrow (y - 4)(y - 8) = 0$

$\Rightarrow y = 4 \text{ or } 8 \Rightarrow 2^x = 2^2 \text{ or } 2^3 \Rightarrow x = 2 \text{ or } 3$

But for $\log_3(2x - 5)$ and $\log_3 (2x - 7/2)$ to be defined

$2^x - 5 > 0$ and $2^x - 7/2 > 0$

$\Rightarrow 2^x > 5$ and $2^x > 7/2$

$\Rightarrow \quad 2^x > 5$

$\Rightarrow x \neq 2$ and therefore $x = 3$.

Sol. 12.

Let a and b be two numbers and $A_1, A_2, A_3, \dots\dots\dots A_n$ be n A. M's between a and b

Then a, $A_1, A_2, \dots\dots A_n$, b are in A. P. There are (n + 2) terms in the series, so that

$a + (n + 1)d = b \Rightarrow d = b - a/n + 1$

$\therefore \quad A_1 = a + b - a/n + 1 = an + b/n + 1$

$\therefore \quad$ ATQ $p = an + b/n + 1$ ..... (1)

The first H. M. between a and b, when nHM's are inserted between a and b can be obtained by replacing a by 1/a and b by 1/b in eq. (1) and then taking its reciptocal.

Therefore, $q = 1/(1/a)n + 1/b/n + 1 = (n + 1)ab/bn + a$

$\therefore q = (n + 1)ab/a + bn$ .... (2)

We have to prove that q cannot lie between p and $(n + 1)^2/(n - 1)^2 p$.

Now, $n + 1 > n - 1 \Rightarrow n + 1/n - 1 > 1$

$\Rightarrow (n + 1/n - 1)^2 > 1$ or $p(n + 1/n - 1)^2 > p$

$\Rightarrow \ p < p(n + 1/n - 1)^2$ …………. (3)

Now to prove the given, we have to show that q is less than p.

For this, let, $p/q = (na + b)\ (nb + a)/(n + 1)^2\ ab$

$\Rightarrow p/q - 1 = n(a^2 + b^2) + ab\ (n^2 + 1) - (n + 1)^2\ ab/(n + 1)^2\ ab$

$\Rightarrow p/q - 1 = n(a^2 + b^2 - 2ab)/(n + 1)^2\ ab$

$\Rightarrow p/q - 1 = n/(n + 1)^2\ (a - b/\sqrt{ab})^2$

$= n/(n + 1)^2\ (\sqrt{a/b} - \sqrt{b/a})^2 \Rightarrow p/q - 1 > 0$

$\Rightarrow$ (provided a and b and hence p and q are +ve)

$p > q$ …….. (4)

From (3) and (4), we get,

$q < p < (n + 1/n + 1)^2\ p$

$\therefore$ q can not lie between p and $(n + 1/n + 1)^2$ p, if a and b are +ve numbers.

ALTERNATE SOLUTION:

After getting equations (1) and (2) as in the previous method, substitute $b = p(n + 1) - an$ [from (1)] in equation (2) to get $aq + nq\ [p(n + 1) - an] = (n + 1)a\ [p(n + 1) - an\ ]$

$\Rightarrow a^2n\ (n + 1) + a\ [q(1 - n^2) - p(n + 1)^2] + npq\ (n + 1) = 0$

$\Rightarrow na^2 - [(n + 1)p + (n - 1)q]\ a + npq = 0$

$\Rightarrow D \geq 0\ \ (\because$ a is real)

$\Rightarrow [n + 1)p + (n - 1)q]^2 - 4n^2pq \geq 0$

$\Rightarrow (n - 1)^2q^2 + \{2(n^2 - 1) - 4n^2\}\ pq + (n + 1)^2p^2 \geq 0$

$\Rightarrow\ q^2 - 2\ n^2 + 1/(n - 1)^2\ pq + (n + 1/n - 1)^2\ p^2 \geq 0$

$\Rightarrow [q - p(n + 1/n - 1)^2\ [q - p] \geq 0$

[On factorizing by discriminent method] $\Rightarrow$ q can not lie between p and $p(n + 1/n - 1)^2$.

Sol. 13.

ATQ we have,

$S_1 = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots \infty$

$S_2 = 2 + 2.\frac{1}{3} + 2\ \left(\frac{1}{3}\right)^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \infty$

. . . .

. . . .

$S_3 = 3 + 3.\frac{1}{4} + 3\left(\frac{1}{4}\right)^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \infty$

$S_n = n + n\,\frac{1}{n+1} + n\left(\frac{1}{n+1}\right)^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots \infty$

$\Rightarrow S_1 = \frac{1}{1-\frac{1}{2}} = 2$ [using $S_\infty = \frac{a}{1-r}$]

$S_2 = \frac{2}{1-\frac{1}{3}} = 3, \quad S_3 = \frac{3}{1-\frac{1}{4}} = 4,$

$S_n = \frac{n}{1-\frac{1}{n+1}} = (n + 1)$

$\therefore S_1^2 + S_2^2 + S_3^2 + \ldots\ldots\ldots\ldots\ldots\ldots S^2_{2n-1}$

$= 2^2 + 3^2 + 4^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + (n + 1)^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + (2n)^2$

NOTE THIS STEP:

$\sum_{r=1}^{2n} r^2 - 1 = \frac{2n(2n+1)(4n+1)}{6} - 1^2$

$= \frac{n\,(2n+1)(4n+1)3}{3}$

Sol. 14.

Since x_1, x_2, x_3 are in A. P. T. Therefore, let $x_1 = a - d$, $x_2 = a$ and $x_3 = a + d$ and x_1, x_2, x_3 are the roots of

$x^3 - x^2 + \beta x + \gamma = 0$

We have $\sum\alpha = a - d + a + a + d = 1$ .... (1)

$\sum\alpha\beta = (a - d)a + a(a + d) + (a - d)(a + d) = \beta$ ..... (2)

$\alpha\beta\gamma = (a - d)a(a + d) = -\gamma$ .... (3)

From (1), we get, $3a = 1 \Rightarrow a = 1/3$

From (2), we get $3a^2 - a^2 = \beta$

$\Rightarrow 3(1/3)^2 - a^2 = \beta \Rightarrow 1/3 - \beta = d^2$

(NOTE : In this equation we have two variables β and d but we have only one equation. So at first sight it looks that this equation cannot be solved but we know that

$d^2 \geq 0 \ \forall \ d \in R$ then ; β can be solved).

$\Rightarrow 1/3 - \beta \geq 0 \qquad \because d^2 \geq 0$

$\Rightarrow \beta \leq 1/3 \Rightarrow \beta \in (-\infty, 1/3]$

From (3), $a(a^2 - d^2) = -\gamma$

$\Rightarrow 1/3\ (1/9 - d^2) = -\gamma \Rightarrow 1/27 - 1/3d^2 = -\gamma$

$\Rightarrow \gamma + 1/27 = 1/3d^2 \Rightarrow \gamma + 1/27 \geq 0$

$\Rightarrow \gamma \geq -1/27 \Rightarrow -\gamma \in [-1/27, \infty)$

Hence $\beta \in (-\infty, 1/3)$ and $\gamma \in [-1/27, \infty]$

Sol. 15.

Solving the system of equations, $u + 2v + 3w = 6$,

$4u + 5v + 6w = 12$ and $6u + 9v = 4$

We get $u = -1/3$, $v = 2/3$, $w = 5/3$

$\therefore u + v + w = 2$, $1/u + 1/v + 1/w = -9/10$

Let r be the common ratio of the G. P., a, b, c, d. Then $b = ar$, $c = ar^2$, $d = ar^3$.

Then the first equation

$(1/u + 1/v + 1/w)\ x^2 + [(b - c)^2 + (c - a)^2 + (d - b)^2]\ x + (u + v + w) = 0$

Becomes

$\frac{9}{10}x^2 + [(ar - ar^2)^2 + (ar^2 - a)^2 + (ar^3 - ar)^2]\ x + 2 = 0$

i.e. $9x^2 - 10a^2 (1 - r)^2 [r^2 + (r + 1)^2 + r^2 (r + 1)^2]x - 20 = 0$

i.e. $9x^2 - 10a^2 (1 - r)^2(r^4 + 2r^3 + 3r^2 + 2r + 1)x - 20 = 0$

i.e. $9x^2 - 10a^2 (1 - r)^2 (1 + r + r^2)^2\ x - 20 = 0$,

i.e. $9x^2 - 10a^2 (1 - r^3)^2\ x - 20 = 0$ .... (1)

The second equation is

$20x^2 + 10(a - ar^3)^2 x - 9 = 0$

i.e., $20x^2 + 10a^2 (1 - r^3)^2 x - 9 = 0$ .... (2)

Since (2) can be obtained by the substitution $x \to 1/x$, equations (1) and (2) have reciprocal roots.

Sol. 16.

Let a – 3d, a – d, a + d and a + 3d be any four consecutive terms of an A. P. with common difference 2d ∵ Terms of A. P. are integers, 2d is also an integer.

Hence $p = (2d)^4 + (a - 3d)(a - d)(a + d)(a + 3d)$

$= 16d^4 + (a^2 - 9d^2)(a^2 - d^2) = a^2 - 5d^2)^2$

Now, $a^2 - 5d^2 = a^2 - 9d^2 + 4d^2$

$= (a - 3d)(a + 3d) + (2d)^2$ = some integer

Thus p = square of an integer.

Sol 17.

Given that $a_1, a_2, \ldots\ldots\ldots\ldots a_n$ are +ve real no's is G. P.

$$\left.\begin{aligned} a_1 &= a \\ a_2 &= ar \\ a_3 &= ar^2 \\ a_n &= ar^{n-1} \end{aligned}\right| \text{ As } a_1, a_2, \ldots\ldots a_n \text{ are +ve} \therefore r > 0$$

A_n is A. M. of $a_1, a_2, \ldots\ldots\ldots, a_n$

$\therefore A_n = a_1 + a_2 + \ldots\ldots\ldots\ldots + a_n/n = a + ar + \ldots\ldots\ldots\ldots + ar^{n-1}/n$

$A_n = \frac{a(1-r)^n}{n(1-r)}$ .... (1) (For $r \neq 1$)

G_n is G. M. of $a_1, a_2, \ldots\ldots\ldots\ldots, a_n$

$\therefore G_n = \sqrt[n]{a_1, a_2 \ldots\ldots\ldots\ldots a_n} = \sqrt[n]{a.ar.ar^2 \ldots\ldots\ldots ar^n - 1}$

$= \sqrt[n]{a^n, r\ \frac{n(n-1)}{2}} = ar^{\frac{(n-1)}{2}}$

$G_n = ar^{\frac{(n-1)}{2}}$ .... (2) ($r \neq 1$)

H_n is H. M. of $a_1, a_2 \ldots\ldots\ldots\ldots a_n$

$\therefore H_n = \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+ \ldots\ldots\ldots\ldots +\frac{1}{a_n}} = \frac{n}{\frac{1}{a}+\frac{1}{ar}+ \ldots\ldots +\frac{1}{ar^{n-1}}}$

$= \frac{n}{\frac{1}{a}\frac{\left(\frac{1}{r^n}-1\right)}{\left(\frac{1}{r}-1\right)}} = \frac{n}{\frac{1}{a}\left(\frac{1-r^n}{r^n}\right).\frac{r}{1-r}}$

$H_n = an\, r^{n-1} (1 - r)/(1 - r^n)$ $(r \neq 1)$. . . . . (3)

We also observe that

$A_n H_n = \frac{a(1-r^n)}{n(1-r)} x \frac{anr^{n-1}(1-r)}{1-r^n} = a^n r^{n-1} = G_n^2$

$\therefore A_n H_n = G_n^2$ . . . . (4)

$\therefore$ Now, G. M. of $G_1, G_2, \ldots\ldots\ldots\ldots G_n$ is

$G = \sqrt[n]{G_1\ G_2 \ldots\ldots\ldots\ldots G_n}$

$G = \sqrt[n]{\sqrt{A_1H_1}\ \sqrt{A_2H_2} \ldots\ldots \sqrt{A_nH_n}}$ [Using (4)

$G = (A_1 A_2 \ldots\ldots\ldots\ldots A_n H_1 H_2 \ldots\ldots\ldots\ldots H_n)^{1/2n}$ . . . . (5)

If r = 1 then

$A_n = G_n = H_n = a$

Also $A_n H_n = G_2^n$

$\therefore$ For r = 1 also, equation (5) holds.

Hence we get

$G = (A_1 A_2 \ldots\ldots\ldots\ldots A_n H_1 H_2 \ldots\ldots\ldots\ldots H_n)^{1/2n}$

Sol. 18.

Clearly $A_1 + A_2 = a + b$

$1/H_1 + 1/H_2 = 1/a + 1/b$

$\Rightarrow H_1 + H_2/H_1 H_2 = a + b/ab = A_1 + A_2/G_1 G_2$

$\Rightarrow G_1 G_2 / H_1 H_2 = A_1 + A_2/H_1 + H_2$

Also $1/H_1 = 1/a + 1/3\ (1/b – 1/a) \Rightarrow H_1 = 3ab/2b + a$

$1/H_2 = 1/a + 2/3(1/b – 1/a) \Rightarrow H_2 = 3ab/2a + b$

$\Rightarrow A_1 + A_2/H_1 + H_2 = \frac{a+b}{3ab(\frac{1}{2b+a}+\frac{1}{2a+b})}$

$= \frac{(2b+a)(2a+b)}{9ab}$

Sol. 19.

Given that a, b, c are in A. P.

$\Rightarrow 2b = a + c$ .... (1)

And a^2, b^2, c^2 are in H. P.

$\Rightarrow \frac{1}{b^2} - \frac{1}{d^2} = \frac{1}{c^2} - \frac{1}{b^2}$

$\Rightarrow (a - b)(a + b)/b^2a^2 = (b - c)(b + c)/b^2c^2$

$\Rightarrow ac^2 + bc^2 = a^2b + a^2c$ [$\because a - b = b - c$]

$\Rightarrow ac(c - a) + b(c - a)(c + a) = 0$

$\Rightarrow (c - a)(ab + bc + ca) = 0$

$\Rightarrow$ either $c - a = 0$ or $ab + bc + ca = 0$

$\Rightarrow$ either $c = a$ or $(a+ c) b + ca = 0$ and then form (i) $2b^2 + ca = 0$

Either $a = b = c$ or $b^2 = a(-c/2)$

i.e. a, b, -c/2 are in G. P. Hence Proved

Sol. 20.

$a_n = 3/4 - (3/4)^2 + (3/4)^3 + \ldots\ldots + (-1)^{n-1} (3/4)^n$

$= \frac{\frac{3}{4}(1-\left(-\frac{3}{4}\right)^n)}{1+\frac{3}{4}} = 3/7\ (1 - (-3/4)^n)$

$B_n = 1 - a_n$ and $b_n > a_n\ \forall\ n \geq n_o$

$\therefore 1 - a_n > a_n \Rightarrow 2a_n < 1$

$\Rightarrow \frac{6}{7}[1 - (-\frac{3}{4})^n] < 1 \Rightarrow -(-\frac{3}{4})^n < 1/6$

$\Rightarrow (-3)^{n+1} < 2^{2n-1}$

For n to be even, inequality always holds. For n to be odd, it holds for $n \geq 7$.

$\therefore$ The least natural no, for which it holds is 6 ($\because$ it holds for every even natural no.)