Solved Examples

Example 1:

Given are two sets A {1, 2, -2, 3} and B = {1, 2, 3, 5}. Is the function f(x) = 2x - 1 defined from A to B?

Solution :

Out of all the ordered pairs, the ordered pairs which are related by the function f(x) = 2x - 1 are {(1, 1), (2, 3), (3, 5) But for (-2) in A, we do not have any value in B. So, this function does not exist from A->B.

Example 2:

A function f is defined as f: N -> N (where N is natural number set) and f(x) = x+2. Is this function ONTO?

Solution :

Since, N = {1, 2, 3, 4, ...} and A = B = N

For : A->B

When x = 1 f(x) = 3

When x = 2 f(x) = 4

So f(x) never assume values 1 and 2. So, B have two elements which do not have any pre-image in A. So, it is not an ONTO function.

Example 3 :

Find the range and domain of the function f(x) = (2x+1)/(x-1) and also find its inverse.

Solution :

This function is not defined for x = 1. So, domain of the function is R -{1}.

Now, for finding the range

Let,(2x+3)/(x-1) = y

=> 2x + 3 = yx - y

=> yx - 2x = y + 3

=> (y - 2)x = y + 3

=> x =(y+3)/(t-2)

So, y cannot assume value 2

Range of f(x) is R - {2}.

Inverse is y =(x+3/x-2) .

Example 4:

Find domain and range of the function f(x) = $(x^2+2x+3)/(x^2-3x+2)$

Solution :

This function can be written as : f(x) =$(x^2+2x+3)/(x-1)(x-2)$.

So, domain of f(x) is R - {1, 2}

For range, let $(x^2+2x+3)/(x^2-3x+2) = y$

=> $(1 - y)x^2 + (2x + 3y) x + 3 - 2y = 0$

for x to be real, Discriminant of this equation must be ≥ 0

$D \geq 0$

=> $(2 + 3y)^2 - 4(1 - y)(3 -2y) \geq 0$

=> $4 + 9y^2 + 12y - 4(3 + 2y^2 - 5y) \geq 0$

=> $y^2 + 32y - 8 \geq 0$

=> $(y + 16)^2 - 264 \geq 0$

=> $y < - 16 - \sqrt{264}$ or $y > - 16 + \sqrt{264}$.

Example 5:

Find the period of following functions

(a) cos3 x + sin 5 x

(b) |cos x| + |sin2 x|

(c) x - |x|.

Solution :

(a) f(x) = cos 3x + sin 5x

period of cos 3x = 2∏/3 and period of sin 5x = 2∏/53

L.C.M. of 2∏/3 and 2∏/5 is 2p

So period of f(x) is 2p.

Note: Let g(x) = cos 3x

g((2∏/3)+x) = cos3 ((2∏/3)+x)

= cos (2∏ + 3x)

= cos 3x

= g(x)

(b) f(x) = |cos x| + |sin2 x|

Period of |cos x| = ∏

Period of |sin 2 x| = ∏/2

So, period of f(x) is ∏

(c) f(x) = x [x]

Let T be the period of this function

=> f(T + x) = f(x)

=> T + x - [T + x] = x - [x]

=> T = [T + x] - [x] .... (1)

=> T = integer - integer

= integer

Let T = 1 (Therefore 1 is the smallest positive integer)

Equation (1) becomes

1 = [1 + x] - [x]

which is true for all x ε R

Period of f(x) is 1.

Example 6:

Show that the inverse of a linear fraction function is always a linear fraction function (except where it is not defined).

Solution:

Let, f(x) = (a+bx)/(c+dx) be the said linear fraction function.

Let at some x it attains value y, so,

(a+bx)/(c+dx) = y

=> a + bx - cy - dxy = 0

=> a - cy + x (b - dy) = 0

=> x = (cy-a)/(b-dy).

Which is again a linear fraction function defined in R except

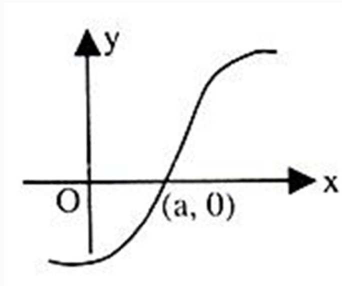
at x = -c/d and y = b/d

and inverse of the given function is, y = (cx-a)/(b-dx).

Example 7:

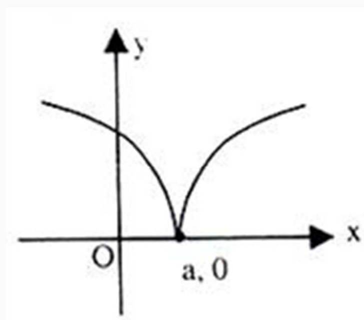
If graph of function f(x) is as shown in the figure given below, then plot the graph of |f(x)|.

f(x) + 1, f(x + 2) and $f^{-1}(x)$

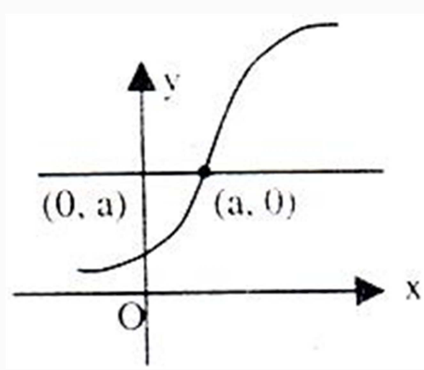


Solution:

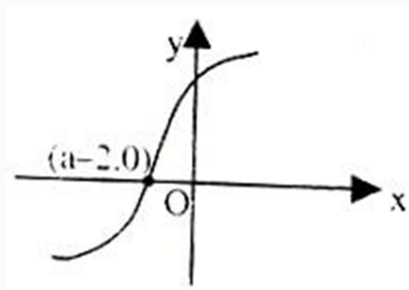
(a) |f(x)| will reflect the graph of f(x) below x axis to the (-) ve y axis side. So the graph will be as shown in the figure given below.



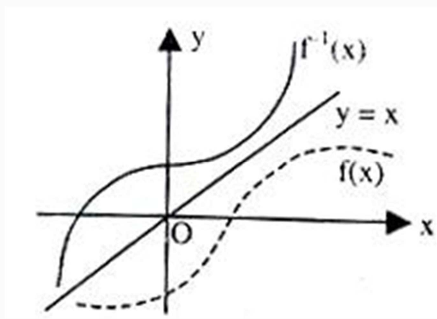
(b) f(x) + 1 will just shift the graph by one unit position up. So the required graph is as shown in the figure given below.



(c) f(x + 2) will shift the graph of f(x) by two units to left, the graph will be as shown in the figure given below.



(d) $f^{-1}(x)$ is obtained by reflection of graph f(x) on the line y = x as shown in the figure given below.



Example 8:

Show that the following functions are even

(a) $f(x) = x^2/(2x^2-1) + x^2/2 + 1$

(b) $f(x) = (a^x+a^{-x})/2$

(c) $f(x) = x^2 - |x|$

Solution:

(a) $f(x) = x^2/(2x^2-1) + x^2/2 + 1$

so, $f(-x) = (-x)^2/(2(-x)^2-1) + (-x)^2/2 + 1$

$= x^2/(2x^2-1) + x^2/2 + 1 = f(x)$

so, f(x) in sum function.

(b) $f(x) = (a^x+a^{-x})/2$

=> $f(-x) = (a^{-x}+a^x)/2 = f(x)$

so, f(x) is even function

(c) $f(x) = x^2 - |x|$

=> $f(-x) = (-x)^2 - |-x| = x^2 - |x| = f(x)$

so, f(x) is even function.

Example 9:

Show that following functions are odd.

(a) $f(x) = (e^x-1)/(e^x+1)$

(b) $f(x) = \log((1-x)/(1+x))$

(c) $f(x) = \sqrt{(1+x+x^2)} - \sqrt{(1-x+x^2)}$

Solution:

(a) $f(x) = (e^x-1)/(e^x+1)$

=> $f(x) = (e^{-x}-1)/(e^{-x}+1) = (1-e^x)/(1+e^x)$

$= -((e^x-1)/(e^x+1)) = -f(x)$

=> $f(-x) = -f(x)$

=> so, f(x) is an odd function

(b) $f(x) = \log((1-x)/(1+x))$

=> $f(-x) = \log((1+x)/(1-x)) \log((1-x)/(1+x))^{-1}$

=> $-\log((1-x)/(1+x))$

=> $f(-x) = -f(x)$

so, f(x) is an odd function.

(c) $f(x) = \sqrt{(1+x+x^2)} - \sqrt{(1-x+x^2)}$

$f(-x) = \sqrt{(x^2-x+1)} - \sqrt{(1+x+x^2)}$

$= -[\sqrt{(1+x+x^2)} - \sqrt{(1-x+x^2)}]$

$f(-x) = -f(x)$

so, f(x) is an odd function

Example 10:

If $f(x) = 1 + x;\ 0 \le x \le 2$

$= 3 - x;\ 2 < x \le 3$

Determine

(a) $g(x) = f(f(x))$

(b) $f(f(f(x)))$

(c) $f([x])$

(d) $[f(x)]$

Where [ ] represents the greatest integer function.

Solution:

(a) g(x) = f(f(x)) = $\begin{cases} 1 + f(x); 0 \le f(x) \le 2 \\ 3 - f(x); 2 < f(x) \le 3 \end{cases}$

First consider the case

f(x) = 1 + x, 0 $\le$ x $\le$ 2

g(f(x)) = $\begin{cases} 1 + 1 + x; 0 \le 1 + x \le 2 \\ 3 - 1 - x; 2 < 1 + x \le 3 \end{cases}$

= $\begin{cases} 2 - x; -1 \le x \le 1 \\ 2 - x; \ 1 < x \le 2 \end{cases}$

Since our considered domain is 0 $\le$ x $\le$ 2. So

g(f(x)) = $\begin{cases} 2 + x; 0 \le x \le 1 \\ 2 - x; 1 < x \le 2 \end{cases}$

For f(x) = 0 3-x, 2 < x $\le$ 3

g(f(x)) = $\begin{cases} 1 + 3 - x; 0 \le 3 - x \le 2 \\ 3 - 3 + x; 2 < 3 - x \le 3 \end{cases}$

= $\begin{cases} 4 - x; 1 \le x \le 3 \\ x; 0 \le x \le 1 \end{cases}$

Since our considered domain is : 2 < x $\le$ 3 so

g(f(x)) = $\{4 - x; 2 < x \le 3\}$

So

g(f(x)) = $\begin{cases} 2 + x; 0 \le x \le 1 \\ 2 - x; 1 < x \le 2 \\ 4 - x; 2 < x \le 3 \end{cases}$

(b) Let 0 $\le$ x $\le$ 1

f(f(f(x)))

= f(2 + x) 2 $\le$ 2 + x $\le$ 3

But we observe that there is no single definition f(f(x)) for this interval.

Therefore we reduce the interval 0 $\le$ x $\le$ 1 to 0 < x $\le$ 1.

Let 0 < x $\le$ 1

f(f(f(x)))

= f(2 + x); 2 < x + 2 < 3

= 3 - (2 + x)

= 1 - x

Let 1 < x $\leq$ 2

= f(2 - x); 0 < 2 - x < 1

= 1 + 2 - x

= 3 - x

Let 2 < x $\leq$ 3

= f(f(f(x)))

= f(4 - x); 1 $\leq$ 4 - x < 2

= 1 + (4 - x)

= 5 - x

f(f(f(x))) = f(f(1)) = f(2) = 3

$$f(f(x))) = \begin{cases} 3; x = 0 \\ 1 - x; 0 < x \leq 1 \\ 3 - x; 1 < x \leq 2 \\ 5 - x; 2 < x \leq 3 \end{cases}$$

(c) f([x])

Let 0 $\leq$ x < 1

f[x] = f(0) = 1

Let 1 $\leq$ x < 2

f[x] = f(1) = 2

Let 2 $\leq$ x < 3

f[x] = f(2) = 3

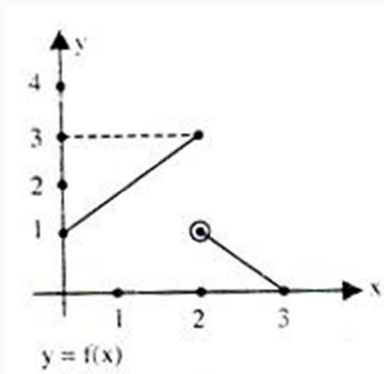
Let x = 3

f([x]) = f(3) = 0

$$\therefore f[x] = \begin{cases} 1, 0 \le x < 1 \\ 2, 1 \le x < 2 \\ 3, 2 \le x < 3 \\ 0, x = 3 \end{cases}$$

(d) [f(x)]

First draw the graph of y = f(x)



Let 0 < x < 1

1 < f(x) < 2 => [f(x)] = 1

Let 1 < x < 2

2 < f(x) < 3 => [f(x)] = 2

Let x = 2

f(x) = 3

[f(x)] = 3

Let 2 < x $\le$ 3

0 < f(x) < 1 => [f(x)] = 0

$$\therefore [f(x)] = \begin{cases} 1, 0 \le x < 1 \\ 2, 1 \le x < 2 \\ 3, x = 2 \\ 0, 2 < x \le 3 \end{cases}$$

Example 11:

If $x^2 + y^2 = 1$

prove that $-\sqrt{2} \le x + y \le \sqrt{2}$.

Solution:

Since, $x^2 + y^2 = 1 \Rightarrow x = \cos\theta,\ y = \sin\theta$

Consider,

$x + y = \cos\theta + \sin\theta$

$= \sqrt{2}((1/\sqrt{2})\sin\theta + (1/\sqrt{2})\cos\theta)$

$= \sqrt{2}\sin((\Pi/4) + \theta)$

Recall : $\sin((\Pi/4)+\theta)$ can take maximum value 1 and minimum value -1.

$\Rightarrow |\sqrt{2}\sin((\Pi/4)+\theta)| \le \sqrt{2}$

$\Rightarrow -\sqrt{2} \le x + y \le \sqrt{2}$. Hence proved.

Example 12:

Check the invertibility of the function $f(x) = (e^x - e^{-x})$; and then find its inverse.

Solution:

We have

$f(x) = e^x - e^{-x};\ x \in R$

$\lim_{x \to \alpha} f(x) = \alpha$

$\lim_{x \to -\alpha} f(x) = -\alpha$

$f'(x) = e^x + e^{-x} > 0$ ∀ ∑€€ R

Therefore f : R -> R

$f(x) = e^x - e^{-x}$ is a bijective function

Therefore f(x) is invertible

Now, $f(x) = y = t - 1/t$ [where $t = e^x$]

$\Rightarrow t^2 - 1 = ty$

$\Rightarrow t^2 - ty - 1 = 0$

$\Rightarrow t = (y+\sqrt{(y^2+4)})/2$ [t cannot be negative]

Now

$t = e^x$

$\Rightarrow e^x = (y+\sqrt{(y^2+4)})/2$

$\Rightarrow x = \log_e ((y+\sqrt{(y^2+4)})/2)$

Therefore Inverse of $y = e^x - e^{-x}$ is $y = \log_e ((x+\sqrt{(x^2+4)})/2)$

Example 13:

If $f(x) = ((1-x)/(1+x))$ $x \neq 1$ and $x \in R$.

Then show that

(i) $f(1/x) = -f(x)$, $x \neq 0$

(ii) $f(f(x)) + f(f(1/x)) \geq 2$ for $x > 0$.

Solution:

$f(x) = (1-x)/(1+x)$, $x \neq 1$ and $x \in R$

$\Rightarrow f(1/x) = (1-(1/x))/(1+(1/x)) = (x-1)/(x+1)$, $x \neq 0$

$\Rightarrow - f(x)$

Now $f(f(x)) = (1-(1-x)/(1+x))/(1+(1-x)/(1+x)) = (2x)/2 = x$

and $f(f(1/x)) = (1-(x-1))/(1+x))/(1+(x-1)/(1+x)) = 2/2x = 1/x$

$\Rightarrow f(f(x)) + f(f(1/x)) = x + 1/x$

$= (\sqrt{x}-(1/\sqrt{x}))^2 + 2$

R.H.S. = 2 + a positive number $\geq$ 2

so $f(f(x)) + f(f(1/x)) \geq 2$

Example 14:

Let A = R - {3},

B = R - {1}, let f: A -> B be defined by f(x) = (x-2)/(x-3). Is f bijective? Give reasons.

Solution:

(a) Let us test the function for injectivity

Let $x_1, x_2 \in A$ and $f(x_1) = f(x_2)$

$\Rightarrow (x_1-2)/(x_1-3) = (x_2-2)/(x_2-3)$

$\Rightarrow x_1 = x_2$

Therefore f is one-one function (injective) ....(1)

(b) Let us test the function for surjectivity

Let y be any arbitrary element of B and suppose there exists an x such that f(x) = y

$(x-2)/(x-3) = y \Rightarrow x = (3y-2)/(y-1)$

since $y \neq 1$, x is real

Also, $x \neq 3$, for if $x = 3$, then $3 = (3y-2)/(y-1)$

or $3y - 3 = 3y - 2 \Rightarrow -3 = -2$, which is false

Thus $x = (3y-2)/(y-1) \in A$ such that $f(x) = y$ i.e. $\forall y \in B$, we have $x \in A$.

and so f is surjective

This proves that f is bijective.

Tricky Examples

Example 15:

Show that if an odd function is invertible, then its inverse is also an odd function.

Solution:

Let y = f(x) be an odd function

Then

$f(-x) = -f(x) = -y$

Since it is invertible, so we can write

$x = g(y)$

Where $g(x) = f^{-1}(x)$

Consider,

$g(-y) = g(-f(x))$

$= g(f(-x)) = -x = -g(y)$

So g(x) is also an odd function.

Example 16:

Sketch the graph of each of the following functions

(a) $f(x) = x^4 - 2x^2 + 3$

(b) $f(x) = 2x/(1+x^2)$

(c) $f(x) = \sin 2x - 2\sin x$.

Solution:

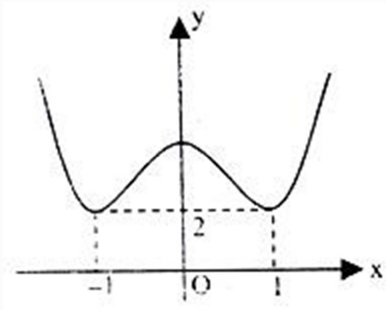
(a) $y = f(x) = x^4 - 2x^2 + 3$

(i) Domain of f(x) is R

(ii) f(x) is even so graph will be symmetrical about y axis.

(iii) $y = x^4 - 2x^2 + 3 = (x^2 - 1)^2 + 2$.

So minimum value of y is at $x^2 = 1 (x = \pm 1)$.



(iv) When x = 0 the value of y = 3

The graph of the function is as shown in fig.

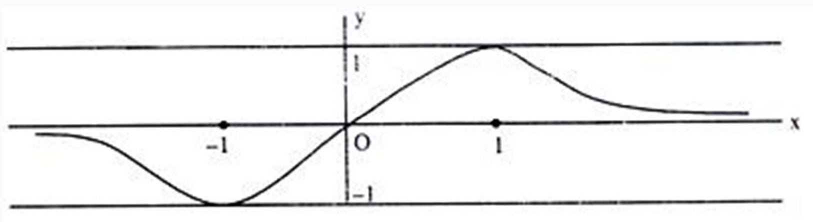
(b) $y = f(x) = 2x/(1+x^2)$.

(i) Domain = R

(ii) f(x) = -f(x), so function is odd the graph is not symmetric about any axis but symmetric about origin.

So it is sufficient to consider only. $x \geq 0$

(iii) y = 0 when x = 0 there is no other point of intersection with co- ordinate axes.



(iv) As $(x - 2)^2 \geq 0$

$\Rightarrow x^2 + 1 \geq 2x$

\

So $2x/(x^2+1) < 1$ and equality holds at x = 1. Also from 0 to 1 the function increases and from 1 to a it decreases. So the graph is as shown in fig.

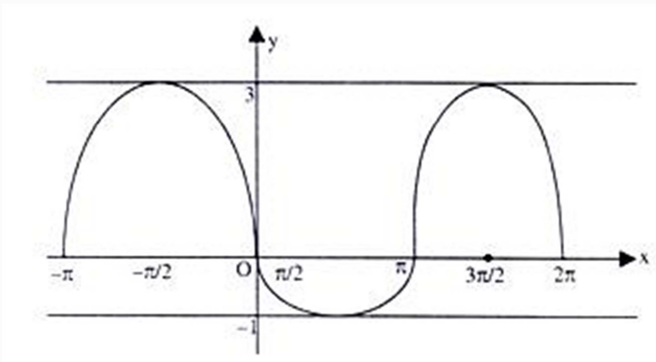
(c) $y = f(x) = \sin^2 x - 2\sin x$

(i) Domain of y is R

(ii) $0 \leq (\sin x - 1)^2 \leq 4$

$\Rightarrow 0 \leq \sin^2 x - 2\sin x + 1 \leq 4$

$\Rightarrow -1 \leq \sin^2 x - 2\sin x \leq 3$



(iii) f(x) has period 2∏ so it is

Sufficient to draw the graph for domain [0, 2∏]

(iv) y = 0 for x = 0, n∏

Note : More about increasing/decreasing we shall study in Module 5.

Example 17:

Solve $(x)^2 = [x]^2 + 2x$

Where [x] represents greatest integer less than or equal to x.

(x) represents integer just greater than or equal to x.

Solution:

Method 1:

Case I :

Let x = n ε I

=> Given equation becomes:

$n^2 = n^2 + 2n$

=> n = 0

Case II:

Let x ε I

i.e. $n \leq x < n + 1$

Given equation becomes:

$(n - 1)^2 = n^2 + 2x$

=> x = n + 1/2, n ε I

Therefore x = 0 or x = n + 1/2; n ε I

Method 2:

Case I :

$x \notin I$

x = [x] + {x}; where {x} represent fraction part of x.

x = (x) - (1 - {x})

$(x + 1 - \{x\})^2 = (x - \{x\})^2 + 2x$ (Using given equation)

=> $(x + 1 - \{x\})^2 + 1 + 2(x - \{x\})^2 = (x - \{x\})^2 + 2x$

=> 1 - 2 {x} = 0

=> {x} = 1/2

$x = n + 1/2, n \varepsilon I$

Also, x = 0, by observation.

Example 18:

Find the set x if the function f:[2, α] -> x where $f(x) = 5 - 4x + x^2$ is bijective.

Solution:

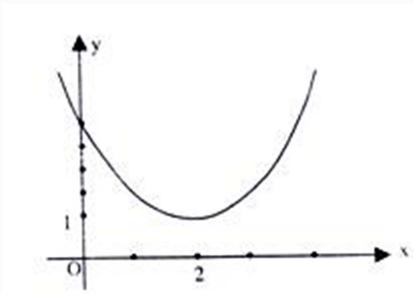
$y = x^2 - 4x + 5$

$= (x - 2)^2 + 1$

When x = 2, y = 1

As $x \varepsilon [2, \alpha)$ then $y \varepsilon [1, \alpha]$

Therefore Set $X \equiv [1, \alpha)$



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