Solved Examples

Example 1:

Given are two sets A \{1, 2, -2, 3\} and B = \{1, 2, 3, 5\}. Is the function \( f(x) = 2x - 1 \) defined from A to B?

Solution:

Out of all the ordered pairs, the ordered pairs which are related by the function \( f(x) = 2x - 1 \) are \{(1, 1), (2, 3), (3, 5)\} But for \(-2\) in A, we do not have any value in B. So, this function does not exist from A->B.

Example 2:

A function \( f \) is defined as \( f: \mathbb{N} \rightarrow \mathbb{N} \) (where \( \mathbb{N} \) is natural number set) and \( f(x) = x+2 \). Is this function ONTO?

Solution:

Since, \( \mathbb{N} = \{1, 2, 3, 4, ........\} \) and A = B = \( \mathbb{N} \)

For : A->B

When \( x = 1 \) \( f(x) = 3 \)

When \( x = 2 \) \( f(x) = 4 \)

So \( f(x) \) never assume values 1 and 2. So, B have two elements which do not have any pre-image in A. So, it is not an ONTO function.

Example 3:

Find the range and domain of the function \( f(x) = (2x+1)/(x-1) \) and also find its inverse.

Solution:

This function is not defined for \( x = 1 \). So, domain of the function is \( \mathbb{R} - \{1\} \).

Now, for finding the range
Let, \( \frac{2x+3}{x-1} = y \)

\[ \Rightarrow 2x + 3 = yx - y \]
\[ \Rightarrow yx - 2x = y + 3 \]
\[ \Rightarrow (y - 2)x = y + 3 \]
\[ \Rightarrow x = \frac{y+3}{y-2} \]

So, \( y \) cannot assume value 2

Range of \( f(x) \) is \( \mathbb{R} - \{2\} \).

Inverse is \( y = \frac{x+3}{x-2} \).

**Example 4:**

Find domain and range of the function \( f(x) = \frac{x^2+2x+3}{x^2-3x+2} \)

**Solution:**

This function can be written as: \( f(x) = \frac{x^2+2x+3}{(x-1)(x-2)} \).

So, domain of \( f(x) \) is \( \mathbb{R} - \{1, 2\} \).

For range, let \( \frac{x^2+2x+3}{x^2-3x+2} = y \)

\[ \Rightarrow (1 - y)x^2 + (2x + 3y) x + 3 - 2y = 0 \]

for \( x \) to be real, Discriminant of this equation must be \( \geq 0 \)

\[ D \geq 0 \]
\[ \Rightarrow (2 + 3y)^2 - 4(1 - y)(3 -2y) \geq 0 \]
\[ \Rightarrow 4 + 9y^2 + 12y - 4(3 + 2y^2 - 5y) \geq 0 \]
\[ \Rightarrow y^2 + 32y - 8 \geq 0 \]
\[ \Rightarrow (y + 16)^2 - 264 \geq 0 \]
\[ \Rightarrow y < - 16 - \sqrt{264} \text{ or } y > - 16 + \sqrt{264}. \]
Example 5:

Find the period of following functions

(a) \( \cos 3x + \sin 5x \)

(b) \( |\cos x| + |\sin 2x| \)

(c) \( x - |x| \).

Solution:

(a) \( f(x) = \cos 3x + \sin 5x \)

period of \( \cos 3x = 2\pi/3 \) and period of \( \sin 5x = 2\pi/5 \)

L.C.M. of \( 2\pi/3 \) and \( 2\pi/5 \) is \( 2\pi \)

So period of \( f(x) \) is \( 2\pi \).

Note: Let \( g(x) = \cos 3x \)

\[
g((2\pi/3)+x) = \cos 3((2\pi/3)+x) \\
= \cos (2\pi + 3x) \\
= \cos 3x \\
= g(x)
\]

(b) \( f(x) = |\cos x| + |\sin 2x| \)

Period of \( |\cos x| = \pi \)

Period of \( |\sin 2x| = \pi/2 \)

So, period of \( f(x) \) is \( \pi \)

(c) \( f(x) = x \lfloor x \rfloor \)

Let \( T \) be the period of this function

\[
=> f(T + x) = f(x)
\]
\[=> T + x - [T + x] = x - [x]\]
\[=> T = [T + x] - [x]\] ........... (1)
\[=> T = \text{integer} - \text{integer}\]
\[= \text{integer}\]

Let \(T = 1\) (Therefore 1 is the smallest positive integer)

Equation (1) becomes
\[1 = [1 + x] - [x]\]

which is true for all \(x \in \mathbb{R}\)

Period of \(f(x)\) is 1.

**Example 6:**

Show that the inverse of a linear fraction function is always a linear fraction function (except where it is not defined).

**Solution:**

Let, \(f(x) = \frac{a+bx}{c+dx}\) be the said linear fraction function.

Let at some \(x\) it attains value \(y\), so,

\[(a+bx)/(c+dx) = y\]

\[=> a + bx - cy - dxy = 0\]

\[=> a - cy + x (b - dy) = 0\]

\[=> x = (cy-a)/(b-dy).\]

Which is again a linear fraction function defined in \(\mathbb{R}\) except

at \(x = -c/d\) and \(y = b/d\)

and inverse of the given function is, \(y = (cx-a)/(b-dx)\).
Example 7:

If graph of function $f(x)$ is as shown in the figure given below, then plot the graph of $|f(x)|$.

$f(x) + 1$, $f(x + 2)$ and $f^{-1}(x)$

Solution:

(a) $|f(x)|$ will reflect the graph of $f(x)$ below $x$ axis to the (-) ve $y$ axis side. So the graph will be as shown in the figure given below.

(b) $f(x) + 1$ will just shift the graph by one unit position up. So the required graph is as shown in the figure given below.
(c) $f(x + 2)$ will shift the graph of $f(x)$ by two units to left, the graph will be as shown in the figure given below.

![Graph](image)

(d) $f^{-1}(x)$ is obtained by reflection of graph $f(x)$ on the line $y = x$ as shown in the figure given below.

![Graph](image)

**Example 8:**

Show that the following functions are even

(a) $f(x) = \frac{x^2}{2x^2-1} + \frac{x^2}{2} + 1$

(b) $f(x) = \frac{a^x + a^{-x}}{2}$

(c) $f(x) = x^2 - |x|

**Solution:**

(a) $f(x) = \frac{x^2}{2x^2-1} + \frac{x^2}{2} + 1$

so, $f(-x) = \frac{(-x)^2}{2(-x)^2-1} + \frac{(-x)^2}{2} + 1$

$= \frac{x^2}{2x^2-1} + \frac{x^2}{2} + 1 = f(x)$
so, f(x) in sum function.

(b) \[ f(x) = \frac{a^x + a^{-x}}{2} \]

\[ \Rightarrow f(-x) = \frac{a^{-x} + a^x}{2} = f(x) \]

so, f(x) is even function

(c) \[ f(x) = x^2 - |x| \]

\[ \Rightarrow f(-x) = (-x)^2 - |-x| = x^2 - |x| = f(x) \]

so, f(x) is even function.

Example 9:

Show that following functions are odd.

(a) \[ f(x) = \frac{e^x - 1}{e^x + 1} \]

(b) \[ f(x) = \log \left( \frac{1-x}{1+x} \right) \]

(c) \[ f(x) = \sqrt{1+x+x^2} - \sqrt{1-x+x^2} \]

Solution:

(a) \[ f(x) = \frac{e^x - 1}{e^x + 1} \]

\[ \Rightarrow f(x) = \frac{e^{-x} - 1}{e^{-x} + 1} = \frac{1-e^x}{1+e^x} \]

\[ = - \left( \frac{e^x - 1}{e^x + 1} \right) = -f(x) \]

\[ \Rightarrow f(-x) = -f(x) \]

\[ \Rightarrow \text{so, f(x) is an odd function} \]

(b) \[ f(x) = \log \left( \frac{1-x}{1+x} \right) \]

\[ \Rightarrow f(-x) = \log \left( \frac{1+x}{1-x} \right) \log \left( \frac{1-x}{1+x} \right)^{-1} \]

\[ \Rightarrow -\log \left( \frac{1-x}{1+x} \right) \]

\[ \Rightarrow f(-x) = -f(x) \]
so, f(x) is an odd function.

(c) \( f(x) = \sqrt{1+x+x^2} - \sqrt{1-x+x^2} \)

\[ f(-x) = \sqrt{x^2-x+1} - \sqrt{1+x+x^2} \]
\[ = -[\sqrt{1+x+x^2} - \sqrt{1-x+x^2}] \]
\[ f(-x) = -f(x) \]

so, f(x) is an odd function

**Example 10:**

If \( f(x) = 1 + x \); \( 0 \leq x \leq 2 \)

\[ = 3 - x \]; \( 2 < x \leq 3 \)

Determine

(a) \( g(x) = f(f(x)) \)

(b) \( f(f(f(x))) \)

(c) \( f([x]) \)

(d) \( [f(x)] \)

Where \([ \ ]\) represents the greatest integer function.

**Solution:**
(a) \[ g(x) = f(f(x)) = \begin{cases} 
1 + f(x), & 0 \leq f(x) \leq 2 \\
3 - f(x), & 2 < f(x) \leq 3
\end{cases} \]

First consider the case
\[ f(x) = 1 + x, \quad 0 \leq x \leq 2 \]
\[ g(f(x)) = \begin{cases} 
1 + 1 + x, & 0 \leq 1 + x \leq 2 \\
3 - 1 - x, & 2 < 1 + x \leq 3
\end{cases} \]
\[ = \begin{cases} 
2 - x, & -1 \leq x \leq 1 \\
2 - x, & 1 < x \leq 2
\end{cases} \]

Since our considered domain is \( 0 \leq x \leq 2 \). So
\[ g(f(x)) = \begin{cases} 
2 + x, & 0 \leq x \leq 1 \\
2 - x, & 1 < x \leq 2
\end{cases} \]

For \( f(x) = 0 \), \( 2 < x \leq 3 \)
\[ g(f(x)) = \begin{cases} 
1 + 3 - x, & 0 \leq 3 - x \leq 2 \\
3 - 3 + x, & 2 < 3 - x \leq 3
\end{cases} \]
\[ = \begin{cases} 
4 - x, & 1 \leq x \leq 3 \\
x, & 0 \leq x \leq 1
\end{cases} \]

Since our considered domain is \( 2 < x \leq 3 \) so
\[ g(f(x)) = \begin{cases} 
4 - x, & 2 < x \leq 3
\end{cases} \]

So
\[ g(f(x)) = \begin{cases} 
2 + x, & 0 \leq x \leq 1 \\
2 - x, & 1 < x \leq 2 \\
4 - x, & 2 < x \leq 3
\end{cases} \]

(b) Let \( 0 \leq x \leq 1 \)
\[ f(f(f(x))) \]
\[ = f(2 + x) \quad 2 \leq 2 + x \leq 3 \]

But we observe that there is no single definition \( f(f(x)) \) for this interval.

Therefore we reduce the interval \( 0 \leq x \leq 1 \) to \( 0 < x \leq 1 \).

Let \( 0 < x \leq 1 \)
\[ f(f(f(x))) \]
\[ f(2 + x); \quad 2 < x + 2 < 3 \]

\[ = 3 - (2 + x) \]

\[ = 1 - x \]

Let \( 1 < x \leq 2 \)

\[ f(2 - x); \quad 0 < 2 - x < 1 \]

\[ = 1 + 2 - x \]

\[ = 3 - x \]

Let \( 2 < x \leq 3 \)

\[ f(f(f(x))) \]

\[ f(4 - x); \quad 1 < 4 - x < 2 \]

\[ = 1 + (4 - x) \]

\[ = 5 - x \]

\[ f(f(f(x))) = f(f(1)) = f(2) = 3 \]

\[ f(f(x)) = \begin{cases} 
3; & x = 0 \\
1 - x; & 0 < x \leq 1 \\
3 - x; & 1 < x \leq 2 \\
5 - x; & 2 < x \leq 3 
\end{cases} \]

(c) \( f([x]) \)

Let \( 0 \leq x < 1 \)

\[ f[x] = f(0) = 1 \]

Let \( 1 \leq x < 2 \)

\[ f[x] = f(1) = 2 \]

Let \( 2 \leq x < 3 \)

\[ f[x] = f(2) = 3 \]
Let $x = 3$

$$f([x]) = f(3) = 0$$

\[
\therefore f[x] = \begin{cases} 
1.0 \leq x < 1 \\
2.1 \leq x < 2 \\
3.2 \leq x < 3 \\
0, x = 3
\end{cases}
\]

(d) $[f(x)]$

First draw the graph of $y = f(x)$

Let $0 < x < 1$

$1 < f(x) < 2 \Rightarrow [f(x)] = 1$

Let $1 < x < 2$

$2 < f(x) < 3 \Rightarrow [f(x)] = 2$

Let $x = 2$

$f(x) = 3$

$[f(x)] = 3$

Let $2 < x \leq 3$

$0 < f(x) < 1 \Rightarrow [f(x)] = 0$
Example 11:

If \( x^2 + y^2 = 1 \)

prove that \(-\sqrt{2} \leq x + y \leq \sqrt{2}\).

Solution:

Since, \( x^2 + y^2 = 1 \) => \( x = \cos \theta, \ y = \sin \theta \)

Consider,

\[ x + y = \cos \theta + \sin \theta \]

\[ = \sqrt{2}((1/\sqrt{2})\sin \theta + (1/\sqrt{2})\cos \theta) \]

\[ = \sqrt{2}\sin((\pi/4) + \theta) \]

Recall : \( \sin((\pi/4)+\theta) \) can take maximum value 1 and minimum value -1.

\[ \implies |\sqrt{2} \sin((\pi/4)+\theta)| \leq \sqrt{2} \]

\[ \implies - \sqrt{2} \leq x + y \leq \sqrt{2}. \quad \text{Hence proved.} \]

Example 12:

Check the invertibility of the function \( f(x) = (e^x - e^{-x}); \) and then find its inverse.

Solution:

We have

\[ f(x) = e^x - e^{-x}; \ x \in \mathbb{R} \]

\[ \lim_{x \to 0} f(x) = 0 \]

\[ \lim_{x \to -\infty} f(x) = -\infty \]

\[ f'(x) = e^x + e^{-x} > 0 \quad \forall \ x \in \mathbb{R} \]
Therefore \( f : \mathbb{R} \rightarrow \mathbb{R} \)

\[ f(x) = e^x - e^{-x} \] is a bijective function

Therefore \( f(x) \) is invertible

Now, \( f(x) = y = t - 1/t \) [where \( t = e^x \)]

\[
\Rightarrow t^2 - 1 = ty \\
\Rightarrow t^2 - ty - 1 = 0 \\
\Rightarrow t = (y+\sqrt{y^2+4})/2 \quad [t \text{ cannot be negative}]
\]

Now

\[ t = e^x \]

\[
\Rightarrow e^x = (y+\sqrt{y^2+4})/2 \\
\Rightarrow x = \log_e ((y+\sqrt{y^2+4}))/2)
\]

Therefore Inverse of \( y = e^x - e^{-x} \) is \( y = \log_e ((x+\sqrt{x^2+4}))/2) \)

Example 13:

If \( f(x) = ((1-x)/(1+x)) \ x \neq 1 \) and \( x \in \mathbb{R} \).

Then show that

(i) \( f(1/x) = -f(x), \ x \neq 0 \)

(ii) \( f(f(x)) + f(f(1/x)) \geq 2 \) for \( x > 0 \).

Solution:

\[ f(x) = (1-x)/(1+x), \ x \neq 1 \] and \( x \in \mathbb{R} \)

\[
\Rightarrow f(1/x) = (1-(1/x))/(1+(1/x)) = (x-1)/(x+1), \ x \neq 0 \\
\Rightarrow - f(x)
\]

Now \( f(f(x)) = (1-(1-x)/(1+x))/(1+(1-x)/(1+x)) = (2x)/2 = x \)
and \( f(f(1/x)) = \frac{1-(x-1)}{(1+x)}/(1+(x-1)/(1+x)) = 2/2x = 1/x \)

\[ \Rightarrow f(f(x)) + f(f(1/x)) = x + 1/x \]

\[ = (\sqrt{x}-(1/\sqrt{x}))^2 + 2 \]

R.H.S. = 2 + a positive number \( \geq 2 \)
so \( f(f(x)) + f(f(1/x)) \geq 2 \)

**Example 14:**

Let \( A = R - \{3\} \),

\( B = R - \{1\} \), let \( f: A \rightarrow B \) be defined by \( f(x) = (x-2)/(x-3) \). Is \( f \) bijective? Give reasons.

**Solution:**

(a) Let us test the function for injectivity

Let \( x_1, x_2 \in A \) and \( f(x_1) = f(x_2) \)

\[ \Rightarrow \left( \frac{x_1-2}{x_1-3}\right) = \left( \frac{x_2-2}{x_2-3}\right) \]

\[ \Rightarrow x_1 = x_2 \]

Therefore \( f \) is one-one function (injective) \( \ldots (1) \)

(b) Let us test the function for surjectivity

Let \( y \) be any arbitrary element of \( B \) and suppose there exists an \( x \) such that \( f(x) = y \)

\[ \frac{x-2}{x-3} = y \Rightarrow x = \frac{(3y-2)}{(y-1)} \]

since \( y \neq 1 \), \( x \) is real

Also, \( x \neq 3 \), for if \( x = 3 \), then \( 3 = \frac{(3y-2)}{(y-1)} \)

or \( 3y - 3 = 3y - 2 \Rightarrow -3 = -2 \), which is false

Thus \( x = \frac{(3y-2)}{(y-1)} \in A \) such that \( f(x) = y \) i.e. \( \forall y \in B \), we have \( x \in A \).
and so \( f \) is surjective
This proves that \( f \) is bijective.

**Tricky Examples**

**Example 15:**
Show that if an odd function is invertible, then its inverse is also an odd function.

**Solution:**

Let \( y = f(x) \) be an odd function

Then
\[
 f(-x) = -f(x) = -y
\]

Since it is invertible, so we can write
\[
 x = g(y)
\]

Where \( g(x) = f^{-1}(x) \)

Consider,
\[
 g(-y) = g(-f(x))
 = g(f(-x)) = -x = -g(y)
\]

So \( g(x) \) is also an odd function.

**Example 16:**

Sketch the graph of each of the following functions

(a) \( f(x) = x^4 - 2x^2 + 3 \)

(b) \( f(x) = 2x/(1+x^2) \)

(c) \( f(x) = \sin 2x - 2 \sin x \)

**Solution:**

(a) \( y = f(x) = x^4 - 2x^2 + 3 \)
(i) Domain of \( f(x) \) is \( \mathbb{R} \)

(ii) \( f(x) \) is even so graph will be symmetrical about \( y \) axis.

(iii) \( y = x^4 - 2x^2 + 3 = (x^2 - 1)^2 + 2. \)

So minimum value of \( y \) is at \( x^2 = 1(x = \pm 1) \).

(iv) When \( x = 0 \) the value of \( y = 3 \)

The graph of the function is as shown in fig.

(b) \( y = f(x) = \frac{2x}{1+x^2} \).

(i) Domain = \( \mathbb{R} \)

(ii) \( f(x) = -f(x) \), so function is odd the graph is not symmetric about any axis but symmetric about origin.

So it is sufficient to consider only. \( x \geq 0 \)

(iii) \( y = 0 \) when \( x = 0 \) there is no other point of intersection with co-ordinate axes.

(iv) As \( (x - 2)^2 \geq 0 \)

\[ \Rightarrow x^2 + 1 \geq 2x \]
So $\frac{2x}{(x^2+1)} < 1$ and equality holds at $x = 1$. Also from 0 to 1 the function increases and from 1 to $a$ it decreases. So the graph is as shown in fig.

(c) $y = f(x) = \sin^2 x - 2\sin x$

(i) Domain of $y$ is $\mathbb{R}$

(ii) $0 \leq (\sin x - 1)^2 \leq 4$

$\Rightarrow 0 \leq \sin^2 x - 2\sin x + 1 \leq 4$

$\Rightarrow -1 \leq \sin^2 x - 2\sin x \leq 3$

(iii) $f(x)$ has period $2\pi$ so it is

Sufficient to draw the graph for domain $[0, 2\pi]$

(iv) $y = 0$ for $x = 0, n\pi$

Note: More about increasing/decreasing we shall study in Module 5.

Example 17:

Solve $(x)^2 = [x]^2 + 2x$

Where $[x]$ represents greatest integer less than or equal to $x$.

($x$) represents integer just greater than or equal to $x$.

Solution:
Method 1:

Case I:

Let \( x = n \in \mathbb{I} \)

\( \Rightarrow \) Given equation becomes:

\[ n^2 = n^2 + 2n \]

\( \Rightarrow n = 0 \)

Case II:

Let \( x \in \mathbb{I} \)

i.e. \( n \leq x < n + 1 \)

Given equation becomes:

\[ (n - 1)^2 = n^2 + 2x \]

\( \Rightarrow x = n + 1/2, n \in \mathbb{I} \)

Therefore \( x = 0 \) or \( x = n + 1/2; n \in \mathbb{I} \)

Method 2:

Case I:

\( x \notin \mathbb{I} \)

\( x = [x] + \{x\}; \) where \( \{x\} \) represent fraction part of \( x \).

\( x = (x) - (1 - \{x\}) \)

\( (x + 1 - \{x\})^2 = (x - \{x\})^2 + 2x \) (Using given equation)

\( \Rightarrow (x + 1 - \{x\})^2 + 1 + 2 (x - \{x\})^2 = (x - \{x\})^2 + 2x \)

\( \Rightarrow 1 - 2 \{x\} = 0 \)

\( \Rightarrow \{x\} = 1/2 \)
\[ x = n + 1/2, \ n \in \mathbb{I} \]

Also, \( x = 0 \), by observation.

**Example 18:**

Find the set \( X \) if the function \( f: [2, \alpha] \to X \) where \( f(x) = 5 - 4x + x^2 \) is bijective.

**Solution:**

\[ y = x^2 - 4x + 5 \]
\[ = (x - 2)^2 + 1 \]

When \( x = 2 \), \( y = 1 \)

As \( x \in [2, \alpha) \) then \( y \in [1, \alpha) \)

Therefore Set \( X \equiv [1, \alpha) \)

---

*Register for online classroom programmes targeting IIT JEE 2010 and 2011. You can also participate in the online tests conducted by askIITians and also give answers in AQAD (A Question A Day) to get a number of benefits in the online live classroom courses. Visit askIITians.com to read online Study material for IIT JEE and AIEEE preparation absolutely free.*