Solved Examples

Example 1:

Find the area common to the curves $x^2 + y^2 = 4x$ and $y^2 = x$.

Solution:

$x^2 + y^2 = 4x$ ........ (i)

$(x - 2)^2 + y^2 = 4$

This is a circle with centre at $(2, 0)$ and radius 2.

$y = \sqrt{4x-x^2}$

$y^2 = x$ ........ (ii)

Parabola with vertex as origin and symmetrical about x-axis. We will find the area above the x-axis and double the area.

The two curves intersect at $4x - x^2 = x$

$x^2 - 3x = 0$

$x(x - 3) = 0$

$x = 0, 3$

$\Rightarrow y = 0, \pm \sqrt{3}$

Therefore pts. are $(0, 0)$ and $(3, \pm \sqrt{3})$
The required area is

\[ A = 2 \left( \int_0^\frac{2a}{b} \sqrt{x} \, dx + \int_\frac{2a}{b}^2 \sqrt{4x - x^2} \, dx \right) \]

\[ = 2 \left[ \left. \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} \right|_0^{\frac{2a}{b}} + \left. \sqrt{4x - x^2} \right|_\frac{2a}{b}^2 \right] \]

\[ = 2 \left[ \frac{2}{3} \cdot 2a^{\frac{3}{2}} + \frac{2a}{b} \sqrt{4a - 2a + \frac{2a}{b}} \right] \]

\[ = 2 \left[ \frac{2}{3} \cdot 2\sqrt{3} + \frac{2a}{b} \sqrt{2\sqrt{3} + 2} \right] \]

\[ = 2 \left[ \frac{2}{3} \cdot 2\sqrt{3} + 2 + 2 \right] \text{ sq. unit} \]

Example 2:

Find the area enclosed between the curve \( y^2 = 4ax \) and parabola \( x^2 = 4by \)

Solution:

For points of intersection solving the two equations:

\[ y^2 = 4ax \]

\[ x^2 = 4by \]

\[ \Rightarrow (x^2/4b)^2 = 4ax \]

\[ x = 0 \text{ or } x = 3\sqrt{(64ab^2)} \]

The area to be determined is shown in the adjacent figure:
⇒ Area = 16/3 ab sq. units

Example 3:

Find the area of the segment cut off from the parabola \( y^2 = 2x \) by the line \( y = 4x - 1 \).

Solution:

The curve and the required area is shown in the adjacent figure.
For points of intersection P and R solving the equations

\[ y^2 = 2x \]

\[ \Rightarrow (4x - 1)^2 = 2x \]

\[ \Rightarrow 16x^2 - 8x + 1 - 2x = 0 \]

\[ \Rightarrow 16x^2 - 10x + 1 = 0 \]

\[ x = \frac{(10\pm\sqrt{(100-64)})}{(2\times16)} \]

\[ x = \frac{(10\pm6)}{32} = \frac{1}{2}, \frac{1}{8} \]

Point P, \( x = \frac{1}{8} \)

Point R, \( x = \frac{1}{2} \)

Point Q, \( x = \frac{1}{4} \) (\( y = 0 \)) in equation \( y = 4x - 1 \)

\[
A = \int_{0}^{1/8} \sqrt{2} x dx + \int_{1/8}^{1/4} (\sqrt{2x} - (4x - 1)) dx + \int_{1/4}^{1/2} (\sqrt{2x} - 4x + 1) dx
\]

\[
A = 2\sqrt{2} \left( \frac{3}{8} \right)^{3/2} + \int_{1/8}^{1/4} (\sqrt{2x} - 4x + 1) dx
\]

\[
= \frac{2\sqrt{2}}{3} \left( \frac{1}{3} \right)^{3/2} + \frac{2\sqrt{2}}{3} \left( \frac{1}{2} \right)^{3/2} - x \left|_{1/8}^{1/4} \right.
\]

\[
= \frac{2\sqrt{2}}{3} \left( \frac{1}{3} \right)^{3/2} + \frac{2\sqrt{2}}{3} \left( \frac{1}{2} \right)^{3/2} - 2 \left( \frac{1}{4} - \frac{1}{64} \right) + \left( \frac{1}{2} - \frac{1}{8} \right)
\]

\[
= \frac{2\sqrt{2}}{3} \left( \frac{1}{3} \right)^{3/2} + \frac{2\sqrt{2}}{3} \left( \frac{1}{2} \right)^{3/2} - \frac{15}{64} + \frac{7}{8}
\]

\[
= \frac{2\sqrt{2} + 9}{22} \text{ sq. unit.}
\]

Example 4:

Find the area included between the curves \( x^2 = 4ay, y = \frac{8a^3}{(x^2 + 4a^2)} \)
Solution:

\[ x^2 = 4ay \] is a parabola systematic about y-axis and passing through origin. Now the curve \( y = \frac{8a^3}{x^2 + 4a^2} \)

(i) When \( x = 0 \), \( y = 2a \) and \( y \neq 0 \)

(ii) This curve is symmetric about y-axis

(iii) \( \frac{dy}{dx} = \frac{8a^3 \left( -2x \right)}{(x^2 + 4a^2)^2} \)

In \( [-\infty, 0] \) \( \frac{dy}{dx} \) is +ve, in \( ]0, \infty[ \) \( \frac{dy}{dx} \) is -ve so at \( x = 0 \) is point of maxima. The curve is as shown in the figure.

\[ \text{Area desired is twice the shaded area. For point of intersection, solving the equation. Simultaneously} \]

\[ \Rightarrow \frac{x^2}{4a} = \frac{8a^3}{(x^2 + 4a^2)} \]

\[ \Rightarrow x^4 + 4a^2x^2 - 32a^4 = 0 \]

\[ x^2 = \frac{-4a^2 \pm \sqrt{16a^4 + 128a^4}}{2} \]

\[ \Rightarrow x^2 = 4a^2 \]

\[ x = \pm 2a \]

The required area OABO is

\[ A = 2 \int_{-2a}^{2a} \left( \frac{8a^3}{x^2 + 4a^2} - \frac{x^2}{4a} \right) dx = \left[ 8a^3 \frac{1}{a} \tan^{-1} \frac{x}{a} - \frac{x^3}{12a} \right]_{-2a}^{2a} \]

\[ = 2 \left[ 8a^2 \tan^{-1} 2 - \frac{8a^2}{12} \right] = 16a^2 \tan^{-1} 2 - \frac{a^2}{2} \text{ sq. units} \]
Example 5:

Determine the area bounded by the curve \( y = x(x - 1)^2 \), the y axis and the line \( y = 2 \).

Solution:

\( y = x(x - 1)^2 \)

The curve is defined everywhere. It is not symmetrical about either axis when \( y = 0 \), \( x = 0 \), or \( x = 1 \) so this curve passes through \((0, 0)\) and \((1, 0)\)

\[
\frac{dy}{dx} = 2x(x - 1) + (x - 1)^2
\]

\[
= (x - 1)(2x + 1 - 1) = (x - 1)(3x - 1)
\]

Critical points \( x = 1 \), \( x = 1/3 \).

The domain is divided into three regions

i.e. \( ]0, 1/3[ \), \( ]1/3, 1[ \), \( ]1, \infty[ \)

in \( ]0, 1/3[ \), \( dy/dx \) is +ve

in \( ]1/3, 1[ \), \( dy/dx \) is −ve

in \( ]1, \infty[ \), \( dy/dx \) is +ve

The curve and the required area are shown in the figure. The area is given by

\[
A = \int_{0}^{1/3} 2dx - \int_{0}^{1} x(x-1)^2 \ dx = 4 - \left[ \frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2} \right]_0^1
\]
Example 6:

Draw the graph of the following function and discuss its continuity and differentiability at \( x = 1 \). Also the area bounded by the curve with x-axis.

**Solution:**

\[ f(x) = \begin{cases} 3^x & -1 < x < 1 \\ 4x & 1 < x < 4 \end{cases} \]

For \([-1, 1]\), \( f(x) \) is an exponential function with index \( > 1 \) so increasing, also \( f'(x) = 3^x \log_3 \) exist everywhere in this domain for \([-1, 1]\) \( f(x) = 4 - x \) is a line with negative slope. \( \frac{dy}{dx} < 0 \) and \( = -1 \) so exist everywhere in this domain. Only point of discontinuity or non-differentiable may be the cusp of two intervals i.e. \( x = 1 \)

\[ f(1) = f(1 - h) = 3^1 = 3 \]

\[ \lim_{h \to 0} f(1 - h) = 3 \]

\[ f(1 + h) = 4 - 1 - h = 3 \]

\[ \lim_{h \to 0} = 3 \] 

\[ f(1 - ) = f(1) = f(1 + h) \]

\[ \text{So function is continuous at } x = 1 \]
Example 7:

Compute the area of the region bounded by the curve $y = e^x \log x$ and $y = (\log x)/e^x$, where $\log e = 1$.

Solution:

Given curves are $y = e^x \log x$ and $y = (\log x)/e^x$

Tracing of curve $y = e^x \log x$

We know $\log x$ is defined for $x > 0$ and hence both curve is defined for $x > 0$.

Tracing of curve $y = e^x \log x$

(i) $y = 0$, $e^x \log e x = 0$

$\Rightarrow \log e x = 0$ or $x = 0$ but $x \neq 0$

$\Rightarrow x = 1$

Hence curve cuts $x$-axis at $(1, 0)$
(ii) Curve is no symmetric about x-axis and y-axis

(iii) By sign scheme, we can conclude

Similarly for curve (ii) \( y = \frac{\log x}{e^x} \)

(i) It cuts x-axis at \((1, 0)\)

(ii) Curve is not symmetric about x-axis and y-axis

(iii) By sign scheme

The two curve intersect at

\[ e^x \log x = \frac{\log x}{e^x} \]

or \((e^{x-1}/e^x) \log x = 0\)

or \(x = 1/e\) or \(x = 1\)

The required area is \(ABCDA = \int_{1/e}^{1} (y_1-y_2) \, dx\)

\[ = \int_{1/e}^{1} e^x \log x - 1/e \int_{1/e}^{1} \frac{\log x}{x} \, dx \]

\[ = e \int_{1/e}^{1} e^x \log x - 1/e \int_{1/e}^{1} \frac{\log x}{x} \, dx \]

\[ \text{.......................... (i)} \]
Let \( \log x = t \)

When \( x = 1, \ t = 0 \)

\( x = 1/e, \ t = -1 \)

\[ \therefore I = \int_{-1}^{0} t \, dt = 2t^2 \bigg|_{-1}^{0} = -1/2 \]

The required area = \( \left| e \left( \frac{3-e^2}{4e^2} \right) - \frac{1}{2} \left( -\frac{5}{2} \right) \right| = \left| \frac{3-e^2}{4e} + \frac{1}{2e} \right| \)

\[ = \frac{5-e^2}{4e} \]

Example 8:

Let \( f(x) = \text{maximum} \ [x^2, (1-x)^2, 2x(1-x)] \) where \( x \in [0, 1] \). Determine the area
of the region bounded by the curve $y = f(x)$ and the lines $y = 0$, $x = 0$, $x = 1$.

**Solution:**

Clearly all curves represent a parabola with different vertex. Let us consider.

(i) $y = x^2$ vertex $(0, 0)$ and at $x = 1$, $y = 1$

(ii) $y = (1 - x)^2$ parabola whose vertex is $(1, 0)$ and at $x = 1$, $y = 0$ and it cuts $x = 0$, at $(0, 1)$

Intersecting point of the above two parabolas is $R(1/2, 1/4)$

$(x^2 = (1-x)^2 \Rightarrow 1 = 2x = 0 \text{ or } x = 1/2 \Rightarrow y = x^2 = 1/4)$

(iii) $y = 2x (1 - x)$ is a parabola

The equation we get $x^2 - x = -1/2 y$

or $(x-1/2)^2 = -1/2 \ (y-1/2)$

So its vertex is $(1/2, 1/2)$. It passes through $(0, 0)$ and $(1, 0)$.

Let this parabola cut $y = (1 - x)^2$ and $y = x^2$ at $D$ and $E$ respectively.

Solving $y = 2x (1 - x)$ and $y = (1 - x)^2$ we get

$2x (1 - x) = (1 - x)^2$

or $(1 - x) (2x - 1 + x) = 0$
\[ \therefore x = 1, \frac{1}{3} \text{ and so } y = 0, \frac{4}{9} \]

\[ \therefore D = \left( \frac{1}{3}, \frac{4}{9} \right) \]

Solving \( y = 2x (1 - x) \) and \( y = x^2 \), we get

\[ 2x (1 - x) = x^2 \]

or \( x(2 - 2x - x) = 0 \)

\[ \therefore x = 0, \frac{2}{3} \text{ and so } y = 0, \frac{4}{9} \]

\[ \therefore E = \left( \frac{2}{3}, \frac{4}{9} \right) \]

\[ \therefore \text{ the equation of the curve} \]

\[ y = f(x) = (1 - x)^2 \quad 0 < x < \frac{1}{3} \]

\[ = 2x (1 - x), \quad \frac{1}{3} < x < \frac{2}{3} \]

\[ = x^2 \quad \frac{2}{3} < x < 1 \]

\[ \therefore \text{ the required area} \]
Example 9:

Find the area of the figure bounded by the curve $|y| = 1 - x^2$

Solution:

The given curve is symmetric about $y$-axis and (as power of $x$ is even) also $x$-axis

Note: Modulus function say $y = |f(x)|$ is symmetric about $y$-axis.

The curve represents two parabolas in the region $x^2 < 1$ or $x^2 > 1$. We trace the curve in first quadrant.
Since curve is symmetric about x-axis and y-axis. Therefore curve would look like

Thus required area is \( PQRS = 4 \text{ Area of OPQO} \)

\[ = 4 \int_{0}^{1}(1-x^2) \, dx \]

\[ = 4 \left[ x-x^3/3 \right]_0^1 = 8/3 \]

**Example 10:**

The tangent drawn from the origin to the curve \( y = 2x^2 + 5x + 2 \) meets the curve at a point whose y-co-ordinate is negative. Find the area of the figure bounded by the tangent between the point of contact and origin, the x-axis and the parabola.

**Solution:**

The given parabola

\[ y = 2x^2 + 5x + 2 \] \( \ldots \) \( (i) \)

In standard form is

\[ y - 2 = 2 (x^2+5/2 \, x) \]

or \( y - 2 + 25/8 = 2 (x+5/4)^2 \)

\[ (y+9/8) = 2 (x+5/4)^2 \]

is a parabola with vertex \((-5/4,-9/8)\)

\[ dy/dx = 4x + 5 \]

\[ \therefore \text{Equation of tangent to (i) at the point (x, y) is} \]

\[ Y - y = 4x + 5 \, (X - x) \] \( \ldots \) \( (ii) \)

If the straight line passes through origin then

\[ -y = (4x + 5) \, (-x) \text{ or } y = 4x^2 + 5x \] \( \ldots \) \( (iii) \)
Subtracting (i) from (iii), we get $2x^2 - 2 = 0$, or $x^2 - 1 = 0$ or $x = \pm 1$

Putting $x = 1$ in (i), we get $y = 9$ and putting $x = 1$, we get $y = 1$.

Therefore the point $P(-1, 1)$ is the point of contact of the tangent drawn from the origin to the parabola (i) meeting at a point whose $y$-co-ordinate is negative.

The equation of the tangent $OP$ is $y = x$.

The required area bounded by the tangent $OP$, the $x$-axis and the parabola

$= \text{the area NOP}$

$= \text{the area of } \triangle OPM - \text{the area PMN}$

\[
= \left| \int_{-1}^{0} x \, dx \right| - \left| \int_{-1}^{-\sqrt{3}/2} (2x^2 + 5x + 2) \, dx \right|
\]

\[
= \frac{1}{2} x^2 \bigg|_{-1}^{0} - \left[ \frac{2x^3}{3} + \frac{5x^2}{2} + 2x \right]_{-1}^{\frac{3}{2}}
\]

\[
= \frac{1}{2} - \left[ \left( -\frac{1}{12} + \frac{5}{8} - 1 \right) - \left( -\frac{2}{3} + \frac{5}{2} + 2 \right) \right]
\]

\[
= \frac{1}{2} - \left( -\frac{1}{12} + \frac{5}{8} - 1 + \frac{2}{3} - \frac{5}{2} - 2 \right) = \frac{1}{2} - \left( -\frac{7}{24} \right)
\]

\[
= \frac{1}{2} + \frac{7}{24} = \frac{5}{24}.
\]