Solved Examples

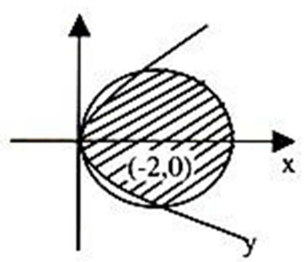
Example 1:

Find the area common to the curves $x^2 + y^2 = 4x$ and $y^2 = x$.

Solution:

$x^2 + y^2 = 4x$ ……… (i)

$(x - 2)^2 + y^2 = 4$



This is a circle with centre at (2, 0) and radius 2.

$y = \sqrt{(4x-x^2)}$

$y^2 = x$ ……… (ii)

Parabola with vertex as origin and symmetrical about x-axis. We will find the area above the x-axis and double the area.

The two curves intersect at $4x - x^2 = x$

$x^2 - 3x = 0$

$x(x - 3) = 0$

$x = 0, 3$

$\Rightarrow y = 0, \pm \sqrt{3}$

Therefore pts. are (0, 0) and $(3, \pm \sqrt{3})$

The required area is

$$A = 2\left(\int_0^3 \sqrt{x}\,dx + \int_3^4 \sqrt{4x - x^2}\,dx\right)$$

$$= 2\left[\frac{2x^{3/2}}{3}\Big|_0^3 + \int_3^4 \sqrt{4 - (x-2)^2}\,dx\right]$$

$$= 2\left[\frac{2}{3}3^{3/2} + \frac{x-2}{2}\sqrt{4x - x^2} + \frac{4}{2}\sin^{-1}\frac{x-2}{2}\Big|_3^4\right]$$

$$= 2\left[2\sqrt{3} + \frac{4}{3}\sin^{-1}1 - \frac{1}{2}\sqrt{3} + 2\sin^{-1}\frac{1}{2}\right]$$

$$= 2\left[\frac{3}{2}\sqrt{3} + 2\frac{\pi}{2} + 2\frac{\pi}{6}\right] = 3\sqrt{3} + \frac{8\pi}{3} \text{ sq. unit}$$

Example 2:

Find the area enclosed between the curve y2 = 4ax and parabola $x^2 = 4by$

Solution:

For points of intersection solving the two equations:

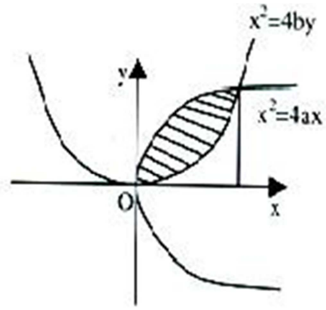
$y^2 = 4ax$

$x^2 = 4by$

$\Rightarrow (x^2/4b)^2 = 4ax$

$x = 0$ or $x = \sqrt[3]{64ab^2}$

The area to be determined is shown in the adjacent figure:



$$\text{Area} = \int_0^{\sqrt[3]{64ab^2}} \left(\sqrt{4ax} - \frac{x^2}{4b}\right) dx$$

$$= \sqrt{4a}\,\frac{2}{3}x^{3/2} - \frac{x^3}{12ab}\Big|_0^{\sqrt[3]{64ab^2}}$$

$$= \frac{2}{3}\sqrt{4a}\,(4ab^2)^{1/2} - \frac{(64ab^2)}{12b}$$

$$= \frac{2}{3}\sqrt{256a^2b^2} - \frac{16}{3}ab$$

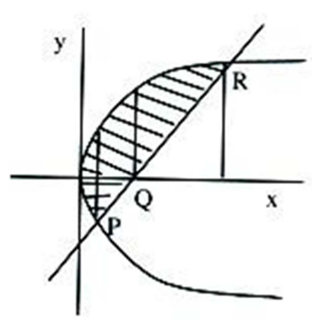
⇒ Area = 16/3 ab sq. units

Example 3:

Find the area of the segment cut off from the parabola $y^2 = 2x$ by the line $y = 4x - 1$.

Solution:

The curve and the required area is shown in the adjacent figure.



For points of intersection P and R solving the equations

$y^2 = 2x$

$\Rightarrow (4x - 1)^2 = 2x$

$\Rightarrow 16x^2 - 8x + 1 - 2x = 0$

$\Rightarrow 16x^2 - 10x + 1 = 0$

$x = (10 \pm \sqrt{(100-64)})/(2 \times 16)$

$x = (10 \pm 6)/32 = 1/2, 1/8$

Point P, $x = 1/8$

Point R, $x = ½$

Point Q, $x = ¼$ ($y = 0$) in equation $y = 4x - 1$

$$A = \int_0^{1/8} \sqrt{2x}\,dx + \int_{1/8}^{1/4} \left(\sqrt{2x} - (4x-1)\right) dx + \int_{1/4}^{1/2} \left(\sqrt{2x} - 4x + 1\right) dx$$

$$A = 2\sqrt{2}\frac{x^{3/2}}{3}\Big|_0^{\frac{1}{8}} + \int_{\frac{1}{8}}^{\frac{1}{2}} (\sqrt{2}x - 4x + 1)\,dx$$

$$\frac{2\sqrt{2}}{3}\left(\frac{1}{8}\right)^{3/2} + 2\sqrt{2}\frac{x^{3/2}}{3} - 2x^2 + x\Big|_{\frac{1}{8}}^{\frac{1}{2}}$$

$$\frac{2\sqrt{2}}{3}\left(\frac{1}{8}\right)^{3/2} + \frac{2\sqrt{2}}{3}\left(\left(\frac{1}{2}\right)^{3/2} - \left(\frac{1}{8}\right)^{3/2}\right) - 2\left(\frac{1}{4} - \frac{1}{64}\right) + \left(\frac{1}{2} - \frac{1}{8}\right)$$

$$\frac{2\sqrt{2}}{3} - \frac{15}{32} + \frac{3}{8}$$

$$\frac{2\sqrt{2}+3}{9} - \frac{15}{32} \text{ sq. unit.}$$

Example 4:

Find the area included between the curves $x^2 = 4ay$, $y = 8a^3/(x^2+4a^2)$

Solution:

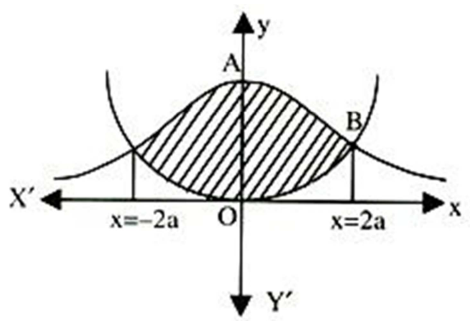
$x^2 = 4ay$ is a parabola systematic about y-axis and passing through origin Now the curve $y = 8a^3/(x^2+4a^2)$

(i) When $x = 0$ $y = 2a$ and $y \neq 0$

(ii) This curve is symmetric about y-axis

(iii) $dy/dx=8a^3 [(-2x)/(x^2+a^2)^2]$

In [– ∞, 0] dy/dx is +v, in]0 ∞[dy/dx is –ve so at x = 0 is point of maxima. The curve is as shown in the



Area desired is twice the shaded area. For point of intersection, solving the equation. Simultaneously

$$\Rightarrow \quad \frac{x^2}{4a} = \frac{8a^3}{(x^2+4a^2)}$$

$$\Rightarrow \quad x^4 + 4a^2x^2 - 32a^4 = 0$$

$$x^2 = \frac{-4a^2 \pm \sqrt{16a^4+128a^4}}{2}$$

$$\Rightarrow \quad x^2 = 4a^2$$

$$x = \pm 2a$$

The required area OABO is

$$A = 2\int_0^{2a}\left(\frac{8a^3}{x^2+4a^2} - \frac{x^2}{4a}\right)dx = \left[8a^3\frac{1}{a}\tan^{-1}x/a - \frac{x^3}{12a}\right]_0^{2a}$$

$$= 2\left[8a^2\tan^{-1}2 - \frac{8a^2}{12}\right] = 16a^2\tan^{-1}2 - \frac{4}{3}a^2 \text{ sq. units}$$

figure.

Example 5:

Determine the area bounded by the curve $y = x(x - 1)^2$, the y axis and the line y = 2.

Solution:

$y = x(x - 1)^2$

The curve is defined everywhere. It is not symmetrical about either axis when y = 0, x = 0, or x = 1 so this curve passes though (0, 0) and (1, 0)

$dy/dx = 2x(x - 1) + (x - 1)^2$

$= (x - 1)(2x + 1 - 1) = (x - 1)(3x - 1)$

Critical points x = 1, x = 1/3.

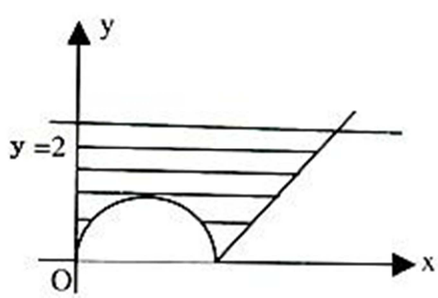
The domain is divided into three regions

i.e.]0, 1/3[,]1/3, 1[,]1, ∞[

in]0, 1/3[, dy/dx is +ve

in]1/3, 1[, dy/dx is −ve

in]1, ∞[, dy/dx is +ve



The curve and the required area are shown in the figure. The area is given by

$A = \int_0^2 2dx - \int_0^2 x(x-1)^2\, dx = 4 - [x^4/4 - 2x^3/3 + x^2/2]_0^2$

= 4 – [16/4-16/3+4/9-0]=10/3 square units

Example 6:

Draw the graph of the following function and discuss its continuity and differentiability at x = 1. Also the area bounded by the curve with x-axis.

Solution:

$f(x) = 3^x$ – 1 < x < 1

= 4x, 1 < x < 4

For [−1, 1], f(x) is an exponential function with index > 1 so increasing, also $f'(x) = 3^x \log 3$ exist everywhere in this domain for [−1, 1] f(x) = 4 – x is a line with negative slope. dy/dx < 0 and = – 1 so exist everywhere in this domain. Only point of discontinuity or non-differentiable may be the cusp of two intervals i.e. x = 1

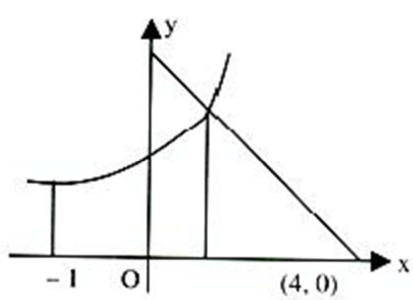
$f(1) = f(1 - h) = 3^1 = 3$

limit h → 0

f(1 + h) = 4 – 1 – h = 3

limit h → 0 – = 3 &threr4;

f(1 –) = f(1) = f(1 + h)



So function is continuous at x = 1

$f'(1-0) = \lim_{h\to 0} \frac{f(1)-f(1-h)}{+h}$

$= \lim_{h\to 0} \frac{3-3^{1-h}}{h} = \lim_{h\to 0} \frac{3(3^{1-h}-1)}{h} = \log 3$

$= (1+0)\, f' = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \frac{41-h-3}{h} = -1$

$f'(1+) \neq f'(1-)$

So function is non-differentiable at x = 1

The area bounded by the curve with the x-axis is

$\int_{-1}^{1} 3^x dx + \int_{1}^{4} (4-x)\, dx = \frac{3^x}{\log 3}\Big|_{-1}^{1} + 4x - \frac{x^2}{2}\Big|_{1}^{4}$

Area $= \frac{8}{3\log 3} + \frac{9}{2}$ square units

Example 7:

Compute the area of the region bounded by the curve y = ex log x and y = (log x)/ex, where log e = 1.

Solution:

Given curves are y = ex log x and y = (log x)/ex

Tracing of curve

We know log x is defined for x > 0 and hence both curve is defined for x > 0.

Tracing of curve y = ex log x

(i) y = 0, ex $\log_e$ x = 0

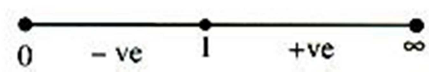
⇒ $\log_e$x = 0 or x = 0 but x ≠ 0

⇒ x = 1

Hence curve cuts x-axis at (1, 0)

(ii) Curve is no symmetric about x-axis and y-axis

(iii) By sign scheme, we can conclude

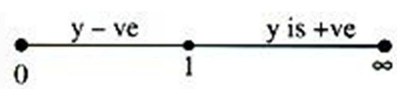


Similarly for curve (ii) y = (log x)/ex

(i) It cuts x-axis at (1, 0)

(ii) Curve is not symmetric about x-axis and y-axis

(iii) By sign scheme



The two curve intersect at

ex log x = (log x)/ex

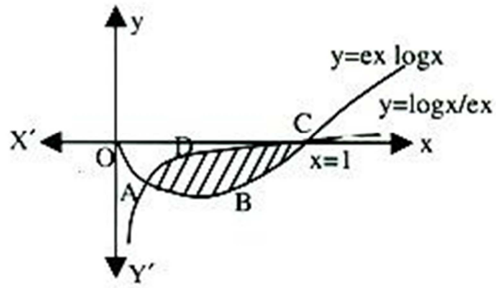
or (ex-1/ex) log x = 0

or x = 1/e or x = 1

The required area is ABCDA = $\int_{1/e}^{1}$ (y1-y2) dx

= $\int_{1/e}^{1}(e^x$ log x (log x)/ex) dx

= e $\int_{1/e}^{1}$x log x -1/e $\int_{1/e}^{1}$ (log x)/x dx .... (i)



$= eI_1 + \frac{1}{e} I_2$

$I_1 = \int_{\frac{1}{e}}^{1} x \log x \, dx = (\log x)\frac{x^2}{2}\Big|_{\frac{1}{e}}^{1} - \int_{\frac{1}{e}}^{1} \frac{1}{x}\frac{x^2}{2} dx$

$= 0 - \frac{1}{2e^2}\log\left(\frac{1}{e}\right) - \frac{1}{4}x^2\Big|_{\frac{1}{e}}^{1}$

$= \frac{1}{2e^2}\log e - \frac{1}{4}\left(1 - \frac{1}{e^2}\right)$

$= \frac{1}{2e^2} - \frac{(e^2-1)}{4e^2} = \frac{3-e^2}{4e^2}$

$I_2 = \int_{\frac{1}{e}}^{1} \frac{\log x}{x} dx$

Let log x = t

When x = 1, t = 0

x = 1/e, t = −1

∴ $I = \int_{-1}^{0} t dt = 2t^2 \, |_{-1}^{0} = -1/2$

The required area $= \left|e\left(\frac{3-e^2}{4e^2}\right) - \frac{1}{e}\left(-\frac{1}{2}\right)\right| = \left|\frac{3-e^2}{4e} + \frac{1}{2e}\right|$

$= \left|\frac{5-e^2}{4e}\right| = \frac{e^2-5}{4e}$

Example 8:

Let f(x) = maximum $[x^2, (1-x)^2, 2x(1-x)]$ where x ∈ [0, 1]. Determine the area of the region bounded by the curve y = f(x) and the lines y = 0, x = 0, x = 1.

Solution:

Clearly all curves represent a parabola with different vertex. Let us consider.

(i) $y = x^2$ vertex (0, 0) and at x = 1, y = 1

(ii) $y = (1 - x)^2$ parabola whose vertex is (1, 0) and at x = 1, y = 0 and it cuts x = 0, at (0, 1)

Intersecting point of the above two parabolas is R(1/2,1/4)

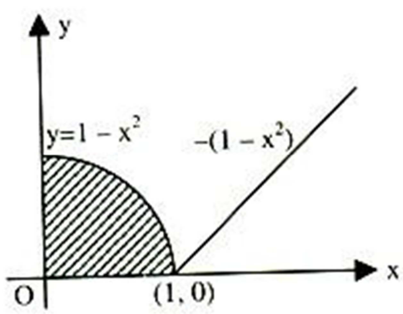
$(x^2=(1-x)^2 \Rightarrow 1=2x=0$ or $x=1/2 \Rightarrow y=x^2=1/4)$

(iii) y = 2x (1 – x) is a parabola

The equation we get $x^2 - x = - 1/2\ y$

or $(x-1/2)^2=-1/2\ (y-1/2)$

So its vertex is (1/2,1/2). It passes through (0, 0) and (1, 0).



Let this parabola cut y = (1 – x)2 and y = x2 at D and E respectively.

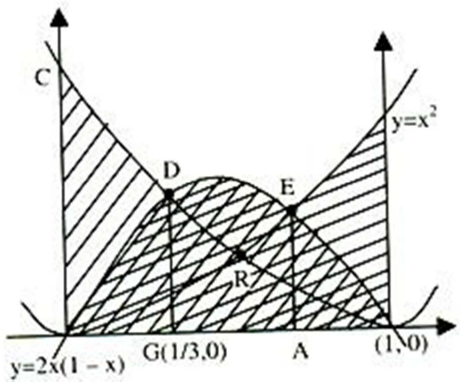
Solving y = 2x (1 – x) and $y = (1 - x)^2$ we get

$2x (1 - x) = (1 - x)^2$

or (1 – x) (2x – 1 + x) = 0

∴ $x = 1, 1/3$ and so $y = 0, 4/9$

∴ $D = (1/3, 4/9)$



Solving $y = 2x(1 - x)$ and $y = x^2$, we get

$2x(1 - x) = x^2$

or $x(2 - 2x - x) = 0$

∴ $x = 0, 2/3$ and so $y = 0, 4/9$

∴ $E = (2/3, 4/9)$

∴ the equation of the curve

$y = f(x) = (1 - x)^2 \qquad 0 < x < 1/3$

$= 2x(1 - x), \qquad 1/3 < x < 2/3$

$= x^2 \qquad 2/3 < x < 1$

∴ the required area

$$= \int_0^{\frac{1}{3}}(1-x)^2\,dx + \int_{\frac{1}{3}}^{\frac{2}{3}} 2x\,(1-x)\,dx + \int_{\frac{2}{3}}^{1} x^2\,\mathrm{dx}$$

$$= \frac{(1-x)^3}{-3}\Big|_0^{\frac{1}{3}} + 2\,\frac{x^2}{2} - \frac{x^3}{3}\Big|_{\frac{1}{3}}^{\frac{2}{3}} + \frac{x^3}{3}\Big|_{\frac{1}{3}}^{\frac{2}{3}} + \frac{x^3}{3}\Big|_{\frac{2}{3}}^{1}$$

$$= -\frac{1}{3}\left[\frac{8}{27} - 1\right] + 2\left\{\frac{1}{2}\left(\frac{4}{9} - \frac{1}{9}\right) - \frac{1}{3}\left(\frac{8}{27} - \frac{1}{27}\right)\right\} + \frac{1}{3}\left(1 - \frac{8}{27}\right)$$

$$= \frac{19}{81} + 2\left(\frac{1}{6} - \frac{7}{81}\right) + \frac{19}{81} = \frac{38}{81} + \frac{13}{81} = \frac{17}{27}$$

Example 9:

Find the area of the figure bounded by the curve $|y| = 1 - x^2$

Solution:

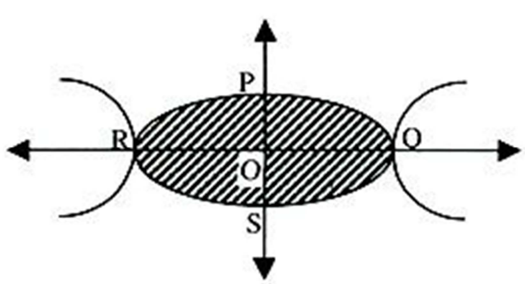
The given curve is symmetric about y-axis and (as power of x is even) also x-axis

Note: Modulus function say y = |f(x)| is symmetric about y-axis.

$$|y| = \begin{cases} y & y \geq 0 \\ -y & y < 0 \end{cases}$$

or, $$\begin{cases} 1 - x^2 & 1 - x^2 \geq 0 & -1 \leq x \leq 1 \\ -(1 - x^2) & 1 - x^2 < 0 & x > 1 \end{cases}$$

The curve represents two parabolas in the region x2 < 1 or x2 > 1. We trace the curve in first quadrant.



Since curve is symmetric about x-axis and y-axis. Therefore curve would look like

Thus required area is PQRS = 4 Area of OPQO

$= 4 \int^0_1 (1-x^2 2)\, dx$

$= 4\ x-x^3/3|_0 1 = 8/3$

Example 10:

The tangent drawn from the origin to the curve $y = 2x^2 + 5x + 2$ meets the curve at a point whose y-co-ordinate is negative. Find the area of the figure bounded by the tangent between the point of contact and origin, the x-axis and the parabola.

Solution:

The given parabola

$y = 2x^2 + 5x + 2$ …… ....(i)

In standard form is

$y - 2 = 2(x^2+5/2\ x)$

or $y - 2 + 25/8 = 2(x+5/4)^2$

$(y+9/8) = 2(x+5/4)^2$

is a parabola with vertex (-5/4,-9/8)

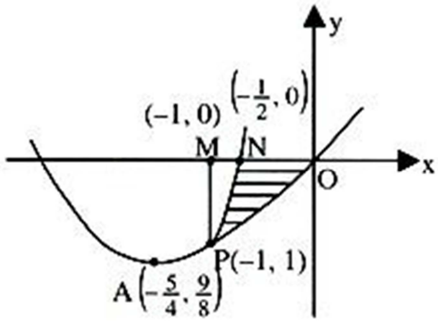
$dy/dx = 4x + 5$

∴ Equation of tangent to (i) at the point (x, y) is

$Y - y = 4x + 5(X - x)$ …………………………………. (ii)

If the straight line passes through origin then

$-y = (4x + 5)(-x)$ or $y = 4x^2 + 5x$ ………………… (iii)



Subtracting (i) from (iii), we get $2x^2 - 2 = 0$, or $x^2 - 1 = 0$ or $x = \pm 1$

Putting $x = 1$ in (i), we get $y = 9$ and putting $x = 1$, we get $y = 1$.

Therefore the point $P(-1, 1)$ is the point of contact of the tangent drawn from the origin to the parabola (i) meeting at a point whose y-co-ordinate is negative.

The equation of the tangent OP is $y = x$.

The required area bounded by the tangent OP, the x-axis and the parabola

= the area NOP

= the area of $\triangle$OPM – the area PMN

$$= \left|\int_{-1}^{0} x\,dx\right| - \left|\int_{-1}^{-1/2}(2x^2 + 5x + 2)\,dx\right|$$

$$= \frac{1}{2}x^2\Big|_{-1}^{0} - \left|2\frac{x^3}{3} + \frac{5x^2}{2} + 2x\Big|_{-1}^{-\frac{1}{2}}\right|$$

$$= \left|-\frac{1}{2}\right| - \left|\left(-\frac{1}{12} + \frac{5}{8} - 1\right) - \left(-\frac{2}{3} + \frac{5}{2} + 2\right)\right|$$

$$= \frac{1}{2} - \left|-\frac{1}{12} + \frac{5}{8} - 1 + \frac{2}{3} - \frac{5}{2} - 2\right| = \frac{1}{2} - \left|-\frac{7}{24}\right|$$

$$= \frac{1}{2} - \frac{7}{24} = \frac{5}{24}.$$