Solved examples of Gravitation

Example 1

The time period of Moon around the Earth is n times that of Earth around the Sun. If the ratio of the distance of the Earth from the Sun to that of the distance of Moon from the Earth is 392, find the ratio of mass of the Sun to the mass of the Earth. (Assume that the bodies revolve in circular orbits)

Solution:

The time period T_e of Earth around Sun of mass M_s is given by

$T_e^2 = 4\Pi^2/GM_e \times r_e^3$, ... (1)

where r_e is the radius of the orbit of Earth around the Sun.

Similarly, time period T_m of Moon around Earth is given by

$T_m^2 = 4\Pi^2/GM_e \times r_e^3$, ... (2)

where r_m is the radius of the orbit of Moon around the Sun.

Dividing equation (1) by (2), we get

$(T_e/T_m)^2 = (M_e/M_s)\ (r_e/r_m)^3$

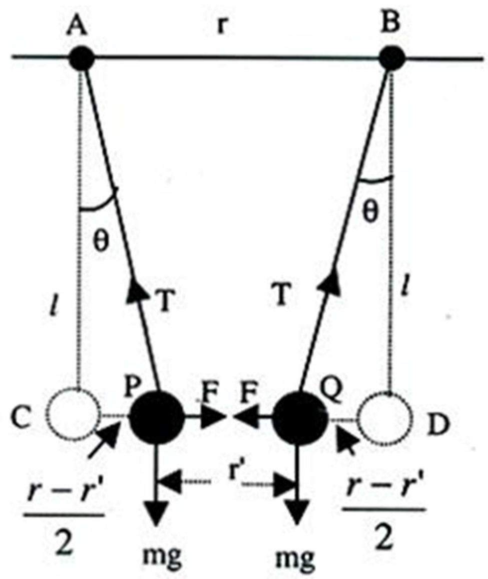
$\therefore$ $(M_s/M_e) = (T_m/T_e)^2 \times (r_e/r_m)^3$ ...... (3)

Substituting the given values, we get

$(M_s/M_e) = (392)^3\ n^2$ Ans.

Example 2

Two balls of mass m each are hung side by side, by two long threads of equal length l. If the distance between upper ends is r, show that the distance r' between the centres of the ball is given by $gr'^2\ (r-r') = 2l\ G\ M$.



Solution:

The situation is shown in figure given above.

Following forces act on each ball

(i) Weight of the ball mg

(ii) Tension in thread T

(iii) Force of Gravitational attraction

$F = G(mm/r^2)$.

As the system is in equilibrium these three forces can be represented by the sides of a triangle say ACP in which AC represents the weight mg, CP represents the force F and PA represents the tension T. Hence

$\tan\ = ((r-r')/2)/l = F/mg$ ...... (2)

But $F/mg = (Gm^2/r'^2)/mg$ ...... (1)

From equations (1) and (2)

$((r-r')/2)/l = (Gm^2/r'^2)/mg$ or $(r-r')mg/2 = lGm^2/r'^2$

$gr'^2\ (r-r') = 2l\ Gm$. (Proved)

Example 3

A planet of mass m moves in an elliptical orbit around the Sun so that its maximum and minimum distances from the Sun are equal to r_1 and r_2 respectively. Find the angular momentum of this planet relative to the centre of the Sun.

Solution:

As the angular momentum of the planet is constant (no external torque is acting on it), we have

$mv_1r_1 = mv_2r_2$

or $v_1r_1 = v_2r_2$

Further, the total energy of the planet is also constant, hence

$-GMm/r_1 + 1/2\, mv_1^2 = -GMm/r_2 + 1/2\, mv_2^2$

where M is the mass of the Sun.

$$\therefore \quad GM\left[\frac{1}{r_2} - \frac{1}{r_1}\right] = \frac{v_1^2}{2} - \frac{v_1^2}{2}$$

$$\text{or} \quad GM\left(\frac{r_1 - r_2}{r_1 r_2}\right) = \frac{v_1^2 r_1^2}{2 r_2^2} - \frac{v_1^2}{2}$$

$$\text{or} \quad GM\left(\frac{r_1 - r_2}{r_1 r_2}\right) = \frac{v_1^2}{2}\left(\frac{r_1^2}{r_2^2} - 1\right) = \frac{v_1^2}{2}\left(\frac{r_1^2 - r_2^2}{r_2^2}\right)$$

$$\therefore \quad v^2{}_1 = \frac{2GM(r_1 - r_2) r_2^2}{r_1 r_2 (r_1^2 - r_2^2)} = \frac{2GM r_2}{r_1(r_1 + r_2)}$$

$$\text{or} \quad v_1 = \sqrt{\left[\frac{2GM r_2}{r_1(r_1 + r_2)}\right]}.$$

Now Angular momentum $= mv_1r_1$

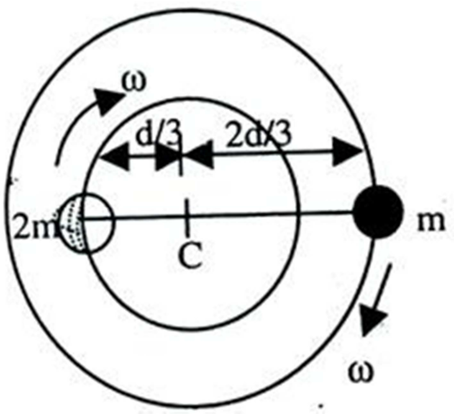
$$= m\sqrt{\left[\frac{2GM r_1 r_2}{(r_1 + r_2)}\right]} \qquad \text{Ans.}$$

Example 4

In a binary star system, two stars (one of mass m and the other of 2m) distance d apart rotate about their common centre of mass. Deduce an expression for the period of revolution. Show that the ratio of their angular momenta about the centre of mass is the same as the ratio of their kinetic energies.

Solution:

The centre of mass C will be at a distance d/3 and 2d/3 from the masses 2m and m respectively as shown in figure 1.10. Both the stars rotate with the same angular velocity ω around C in their respective orbits. Here the gravitational force acting on each star due to other supplies the necessary centripetal force.



Gravitational force on each star = $G(2m)m/d^2$.

Centripetal force of star (mass m)

$= m r \omega^2 = m(2d/3)\omega^2$

$\therefore \quad G(2m)m/d^2 = m(2d/3)\omega^2$

$\omega = \sqrt{(3Gm/d^3)}$

$\therefore T = 2\Pi/\omega = 2\Pi\sqrt{(d^3/3Gm)}$ Ans.

Ratio of angular momentum

$=(I\omega)_{big}/(I\omega)_{small} = (2m(d/3)^2)/(m(2d/3)^2) = 1/2$ Ans.

This is same as that of their angular momenta.

Example 5

Imagine a planet whose diameter and mass are both one half of those of Earth. The day's temperature of this planet's surface reaches upto 800K. Find whether oxygen molecules are possible in the atmosphere of this planet.

Solution:

Escape velocity, $v_e = \sqrt{2GM/R}$

Let v_p = escape velocity on the planet.

v_e = escape velocity on the Earth.

$\therefore \quad v_p/v_e = \sqrt{(M_P/R_P \times R_E/M_E)} = \sqrt{(1/2 \times 2/1)} = 1$

$\therefore \quad v_p = v_e = 11.2$ km/s.

From kinetic theory of gases

$v_{rms} = \sqrt{(3RT/M)} = \sqrt{(3NKT/M)} = \sqrt{(3NKT/Nm)}$

where N = Avogadro's number

m = mass of oxygen molecule

K = Boltzmann constant

$v_{rms} = \sqrt{3RT/m}$

$= \sqrt{((3 \times 1.38 \times 10^{-23} \times 800)/(5.3 \times 10^{-26}))}$ $(m = 5.3 \times 10^{-26}$ kg).

$v_{r.m.s} = 0.79$ km/s Ans.

As v_{rms} is very small compared to escape velocity on the planet, molecules cannot escape from the surface of the planet's atmosphere.

Example 6

Halley's comet has a period of 76 years and in 1986 it had a distance of closest approach to the Sun equal to $8.9 \times 10^{+10}$ m. What is the comet's farthest distance from the Sun if mass of Sun is 2×10^{30} kg and $G = 6.67 \times 10^{-11}$ MKS units.

Solution:

Whenever a heavenly body revolves round another body it follows an elliptical path having a distance o closest and farthest approach.

For these orbits

$T_2 = (4\Pi^2/GM)a^3$, where a and T are average distance of comet from Sun and time period of revolution respectively.

$\therefore \quad a = [T^2GM/4\Pi^2]^{1/3}$

or, $\quad a = [((76\times3.15\times10^7)^2\times 6.67\times10^{-11}\times2\times10^{30})/4\Pi^2]^{1/3}$

$\approx 2.7 \times 10^{12}$ m.

But for ellpse,

$2a = r_{min.} + r_{max.}$

(where a is the semi major axis of the ellipse)

or, $\quad r_{max.} = 2a - r_{min.}$

$r_{max} = 2 \times 2.7 \times 10^{12} - 8.9 \times 10^{10}$

$r_{max} \approx 5.3 \times 10^{12}$ m. Ans.

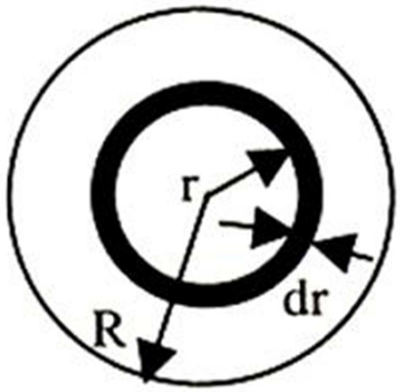
Example 7

A uniform sphere has a mass M and radius R. Find the pressure P inside the sphere caused by gravitational compression, as a function of distance r from its centre.

Solution:

Density of the given sphere, $\rho = M/(4/3\Pi R^3)$

Let us consider a layer of thickness 'dr' at a distance r from the centre. The part of sphere inside this spherical shell will attract this with a net force but the outer part would not have any effect on this as inside a hollow spherical shell force on any particle is zero.



Mass of sphere of radius r, $M_1 = 4/3\Pi R^3\ \rho$

$\therefore \qquad M_1 = M/R^3.r^3$

Mass of the layer of thickness dr is

$$dm = 4\Pi r^2.dr.\rho$$

$$= 4\Pi r^2.dr.M/(4/3\Pi R^3)$$

$$= 3M/R^3 dr\ .\ r^2$$

$\therefore$ Force on small layer due to M_1

$$dF = (GM_1.dm)/r^2 = (GM.r^3.3M.r^2)/(R^3.r^2.R^3)\ dr$$

$$= 3GM^2/R^6\ dr.r^3$$

Thus pressure on the layer of mass 'dm'

$$dp = dF/4\Pi r^2$$

$$= 3GM^2/R^6\ .\ (r^3.dr)/r^2 = 3GM^2/(4\Pi R^6).rdr$$

$\therefore$ Total pressure due to mass extending from r to R is

$$p = \int_r^R dp = 3GM^2/R^6 \int_r^R rdr$$

$$= 3GM^2/(8\Pi R^6)\ .\ [r^2/2]_r^R$$

$$= 3GM^2/(8\Pi R^6)\ (R^2 - r^2)$$

$\therefore \quad p = 3GM^2/(8\Pi R^4)(1-r^2/R^2)$ Ans.

Example 8

Assuming the Earth to be a sphere of uniform density, calculate the energy needed to completely disassemble it against gravitational pull amongst its constituent particles given product of mass and radius of Earth = 2.5×10^{31} kgm.

Solution:

Let us suppose that Earth is made of a infinite number of very thin concentric spherical shells. It can be completely disassembled by removing these shells one by one.

Let us study Earth when only a sphere of radius x is left and find the energy required to remove a shell of thickness dx.

Now Potential of this sphere of radius x is $u = Gm_s/x$

[m_s = mass of sphere = $4/3\ \Pi x^3 \rho$ where ρ = density of Earth = $M/(4/3\Pi R^3)$ where M, R are mass and radius of Earth respectively].

Now work done (dw) to remove a shell of thickness dx would be equal to the change in potential energy = $Gm_s dm/x$,

where dm = mass of shell = $4\Pi x^2\ dx\rho$.

$\Rightarrow dW = (4/3\ x^3\ \rho)(4\Pi x^2\ dx\rho)/x = 16/3\ G\Pi^2\rho^2 x^4\ dx$..

$W = \int dW = \int_0^R 16/3 G\Pi^2\ \rho^2\ x^4\ dx = 16/15\ G\Pi^2\ [m/((4/3)\Pi\ R^3\ )]^2\ R^5$..

Which, on substituting the given value, is $3/5\ GM^2/R = 3/5\ g\ MR$

$= 1.5 \times 10^{32}$ J.

Example 9

A satellite revolving close to the surface of Earth from west to east, appears over a certain point at the equator every 11.6 hours. If radius of Earth is 6400 km. calculate the mass of Earth.

Solution:

Since the rotation of Earth is also west to east the apparent angular velocity of the satellite is equal to

$\omega_{SE} = \omega_S - \omega_e$,

Where ω_S is the angular velocity of the satellite w.r.t. an irrotational frame of reference fixed on the axis of Earth's rotation and ω_e is the angular velocity of Earth w.r.t. the same frame of reference.

$\therefore \quad \omega_S = \omega_{SE} + \omega_e$

now $(GM_e m)/R^2 \; m R\omega^2$,

where R = radius of Earth as the satellite is rotating close to its surface and M_e is the mass of Earth and m is the mass of satellite.

$\therefore \quad M = (R^3 \omega^2)/G = R^3/G \, [2\Pi/T_{SE} + 2\Pi/T_E]^2$.

where T_{SE} is the apparent time period of satellites revolution and T_E = 24 hrs. is the time period of Earth's rotation.

Substituting the values

$M = 6.0 \times 10^{24}$ kg.