Class: XII
Subject: Mathematics
Topic: Inverse Circular Functions
No. of Questions: 20
Duration: 60 Min

Q1. Find the range of function, $f(x) = \cot^{-1}x + \sec^{-1} x + \text{cosec}^{-1}x$.

(a) $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

(b) $\left(\frac{\pi}{2}, \frac{3\pi}{4}\right] \cup \left[\frac{5\pi}{4}, \frac{3\pi}{2}\right)$

(c) $\left[\frac{\pi}{2}, \pi\right) \cup \left(\pi, \frac{3\pi}{2}\right]$

(d) $\left(\frac{\pi}{2}, \pi\right) \cup \left(\pi, \frac{3\pi}{2}\right)$

Solution: B

Domain of f(x) is $|x| \geq 1$

From the graph for $x \geq 1$ f (x) can attain values from $\left(\frac{\pi}{2}, \frac{3\pi}{4}\right]$

Also for $x \leq -1$ f (x) can attain values from $\left[\frac{5\pi}{4}, \frac{3\pi}{2}\right)$

Q2. There exists a positive real number x satisfying $\cos(\tan^{-1}x) = x$. the value of $\cos^{-1}\left(\frac{x^2}{2}\right)$ is

(a) $\frac{\pi}{10}$

(b) $\frac{\pi}{5}$

(c) $\frac{2\pi}{5}$

(d) $\frac{4\pi}{5}$

Solution: C

Let $\tan^{-1}(x) = 0 \quad \Rightarrow \quad x = \tan\theta$

$\text{Cos}\theta = x$ (given)

$$\frac{1}{\sqrt{1+x^2}} = x$$

$$x^2(1+x^2) = 1 \Rightarrow \quad x^2 = \frac{-1\pm\sqrt{5}}{2} \text{ (}x^2\text{ can not be – ve)}$$

$$\Rightarrow \quad x^2 \frac{\sqrt{5}-1}{2} \quad \Rightarrow \quad \frac{x^2}{2} = \frac{\sqrt{5}-1}{4}$$

$$\cot^{-1}\left(\frac{\sqrt{5}-1}{4}\right) = \cot^{-1}\left(\sin\frac{\pi}{10}\right) = \cot^{-1}\left(\cos\frac{2\pi}{5}\right) = \frac{2\pi}{5}.$$

Q3. $2\cot\left(\cot^{-1}(3) + \cot^{-1}(7) + \cot^{-1}(13) + \cot^{-1}(21)\right)$ has the value equal to

(a) 1 (b) 2

(c) 3 (d) 4

Solution: C

Consider $\sum \cot^{-1}(n^2+n+1)$ where n = 1, 2, 3, 4

$$\sum \tan^{-1}\frac{1}{1+n(n+1)}$$

$$\sum \tan^{-1}\left(\frac{(n+1)-n}{1+n(n+1)}\right) = \tan^{-1}(n+1) - \tan^{-1}(n)$$

∴ $S = T_1 + T_2 + T_3 + T_4$

$= (\tan^{-1}(2) - \tan^{-1}(1)) + (\tan^{-1}(3) - \tan^{-1}(2)) + (\tan^{-1}(4) - \tan^{-1}(3)) + (\tan^{-1}(5) - \tan^{-1}(4))$

$= \tan^{-1}(5) - \tan^{-1}(1) = \tan^{-1}\frac{5-1}{1+5} = \tan^{-1}\frac{2}{3}$]

∴ $2\cot\left(\cot^{-1}\frac{3}{2}\right) = 2.\frac{3}{2} = 3 \quad \Rightarrow$ (c)

Q4. Let $\cos^{-1}(x) + \cos^{-1}(2x) + \cos^{-1}(3x) = \pi$. If x satisfies the cubic $ax^3 + bx^2 + cx - 1 = 0$, then $(a + b + c)$ has the value equal to

(a) 24 (b) 25

(c) 26 (d) 27

Solution C

$\cos^{-1}(2x) + \cos^{-1}(3x) = \pi - \cos^{-1}(x) = \cos^{-1}(-x)$

$\cos^{-1}[(2x)(3x) - \sqrt{1-4x^2}\sqrt{1-9x^2}] = \cos^{-1}(-x)$

$6x^2 - \sqrt{1-4x^2}.\sqrt{1-9x^2} = -x$

$(6x^2 + x)^2 = (1 - 4x^2)(1 - 9x^2) \Rightarrow \quad x^2 + 12x^3 = 1 - 13x^2 \Rightarrow \quad 12x^3 + 14x^2 - 1 = 0$

$\therefore \quad a = 12; b = 14; c = 0 \Rightarrow \quad a + b + c = 26$

Q5. The value of $\tan^{-1}\frac{4}{7} + \tan^{-1}\frac{4}{19} + \tan^{-1}\frac{4}{39} + \tan^{-1}\frac{4}{67} + \ldots\ldots\infty$ equals

(a) $\tan^{-1}1 \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$ (b) $\tan^{-1}1 + \cot^{-1}3$

(c) $\cot^{-1}1 \cot^{-1}\frac{1}{2} + \cot^{-1}\frac{1}{3}$ (d) $\cot^{-1}1 + \tan^{-1}3$

Solution B

Let

$$S = 7 + 19 + 39 + 67 + \ldots\ldots + T_n$$
$$S = 0 + 7 + 19 + 39 + \ldots\ldots\ldots\ldots + T_{n-1} + T_n$$

$(subtracting) \quad - \quad - \quad - \quad - \quad - \quad - \quad -$

$$T_n = 7 + 12 + 20 + 28 + \ldots\ldots + (T_n - T_{n-1})$$

$$= 7\frac{(n-1)}{2}[24 + 8(n-2)] = 4n^2 + 3$$

$$\therefore \quad T_n' = \tan^{-1}\frac{4}{4n^2+3} = \tan^{-1}\frac{1}{n^2+\frac{3}{4}} = \tan^{-1}\frac{1}{1+\left(n^2-\frac{1}{4}\right)}$$

$$= \tan^{-1}\left[\frac{\left(n+\frac{1}{2}\right)-\left(n-\frac{1}{2}\right)}{1+\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}\right] = \tan^{-1}\left(n+\frac{1}{2}\right)-\tan^{-1}\left(n-\frac{1}{2}\right)$$

Hence $S\infty = \sum_{n=1}^{\infty} T_n{}' = \frac{\pi}{2} - \tan^{-1}\frac{1}{2} = \tan^{-1}1 + \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} - \tan^{-1}\frac{1}{2} = \tan^{-1}1 + \cot^{-1}3$

Q6. The value of $\tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}} + \tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}} + \tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}$ is :

(Where a, b, c > 0)

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$

(c) π (d) 0

Solution C

$$S = \tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}} + \tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}} + \tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}$$

$$X + y + z = \sqrt{a+b+c}\left(\sqrt{\frac{a}{bc}}+\sqrt{\frac{b}{ca}}+\sqrt{\frac{c}{ab}}\right) = \sqrt{a+b+c}\left(\frac{a+b+c}{\sqrt{abc}}\right) = \frac{(a+b+c)^{3/2}}{(abc)^{1/2}}$$

$$X\,y\,z = \sqrt{\frac{a(a+b+c)}{bc}}\cdot\sqrt{\frac{b(a+b+c)}{ca}}\cdot\sqrt{\frac{c(a+b+c)}{ab}} = \frac{(a+b+c)^{3/2}}{(abc)^{1/2}}$$

∴ $x + y + z = x\,y\,z \Rightarrow S = \pi \Rightarrow$ (c)

Q7. If $\tan^{-1}x = \frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right)$, then x =

(A) $\tan 3\theta$ B) $2\tan\theta$

(C) $\frac{1}{3}\tan\theta$ (D) $3\cot\theta$

Solution C

If t = tanθ then sin2θ= $\frac{2t}{1+t^2}, \cos 2\theta = \frac{1-t^2}{1+t^2}$

$$\therefore \frac{3\sin 2\theta}{5+4\cos 2\theta} = \frac{3.2t}{5(1+t^2)+4(1-t^2)} = \frac{6t}{9+t^2}$$

$$= \frac{2.t/2}{1+(t/3)^2} = \frac{2\tan a}{1+\tan^2 a} = \sin 2a$$

Where $\frac{1}{3} = \tan a$

$$\frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right\} = a = \tan^{-1}\left(\frac{t}{3}\right) = \tan^{-1}\left(\frac{1}{3}\tan\theta\right)$$

$$\Rightarrow \tan^{-1} x = \tan^{-1}\left(\frac{1}{3}\tan\theta\right) \therefore x = \frac{1}{3}\tan\theta$$

Paragraph for question nos. 8 to 10

Let f: A ⟶ B be an onto function defined as f(x) = $\frac{\sin^{-1}x + \tan^{-1}x}{\cos^{-1}x + \cot^{-1}x}$

Q8. If the minimum and maximum value of f(x) be m and respectively then the value of $\frac{M}{m}$ is

(a)-1 (b) -4

(c) -7 (c) infinite

Solution

$$We\ have\ f(x) = \frac{\frac{\pi}{2}\cos^{-1}x + \frac{\pi}{2}\cot^{-1}x}{\cos^{-1}x + \cot^{-1}x} = \frac{\pi - \cos^{-1}x - \cot^{-1}x}{\cos^{-1}x + \cot^{-1}x} = \frac{\pi}{\cos^{-1}x + \cot^{-1}x}$$

$$Clearly\ A = [-1, 1].\ So\ in\ [-1, 1],\ \cos^{-1}x \in [0, \pi] \text{ and } \cot^{-1}x \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$$

$$\therefore \cos^{-1}x + \cot^{-1}x = \left[\frac{\pi}{4}, \frac{7\pi}{4}\right] \forall x \in A$$

$$Also\ \cos^{-1}x + \cot^{-1}x\ is\ strictly\ decreasin\,g\ on\ A$$

$$\therefore m = f(\mathrm{x})_{\min} = \frac{\pi}{(\cos^{-1} x + \cot^{-1} x)_{\max}} - 1 = \frac{\pi}{\cos^{-1}(-1) + \cot^{-1}(-1)} - = \frac{-3}{2}$$

$$and\ M = f(\mathrm{x})_{\max} = \frac{\pi}{(\cos^{-1} x + \cot^{-1} x)_{\min}} - 1 = \frac{\pi}{\cos^{-1} 1 + \cot^{-1} 1} - 1 = 3$$

$$Hence\ \frac{M}{m} = \frac{3}{\left(-\frac{3}{7}\right)} = -7$$

Q9. Let g: B ⟶ A be a function such that g(f(x)) = x ∀ x ∈ A and f(g(x)) = x ∀ x ∈ B, then the value of g (3) is

(a) -1 (b) 0

(c) 0.5 (d) 1

Solution

Clearly g(x) will be inverse off(x).

As f (1) = 3 ⟹ f^{-1} (3) = 1

Hence g (3) = 1

Q10. The number of solutions of the equation f (x^3 +14x^2 +13x – 5) = f (1 – x^2 + x^3) is

(a) 0 (b) 1

(c) 2 (d) 3

Solution

As f(x) is one-one function, so f(x^3 + 14x^2 + 13x – 5) = f(1 - x^2 + x^3)

⟹ x^3 + 14x^2 + 13x – 5 = 1 - x^2 + x^3 ⟹ 15x^2 +13x – 6 = 0

⟹ 15x^2 +18x – 5x – 6 = 0 ⟹ (3x - 1)(5x + 6) = 0

⟹ x = $\frac{1}{3}, \frac{-6}{5}$ but x ϵ [-1, 1]

Hence x = $\frac{1}{3}$ only.

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Q11. . If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$, then the value of $\sum \frac{(x^{101}+y^{101})(x^{202}+y^{202})}{(x^{303}+y^{303})(x^{404}+y^{404})}$ is:

(A) 1 (B) 2

(C) 3 (D) 4

Solution: C We have, $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$

as we know that

$\sin^{-1}x \le \pi/2 \quad x \in [-1,1]$

Hence above result is possible only when.

$\sin^{-1}x = \pi/2$, $\sin^{-1}y = \pi/2$, $\sin^{-1}z = \pi/2$

$\Rightarrow$ $x = 1, y = 1, z = 1$

$\therefore$ $\sum \frac{(x^{101}+y^{101})(x^{202}+y^{202})}{(x^{303}+y^{303})(x^{404}+y^{404})}$.

$\Rightarrow$ $\frac{(x^{101}+y^{101})(x^{202}+y^{202})}{(x^{303}+y^{303})(x^{404}+y^{404})} + \frac{(y^{101}+z^{101})(y^{202}+z^{202})}{(y^{303}+z^{303})(y^{404}+z^{404})}$

$+ \frac{(z^{101}+x^{101})(z^{202}+x^{202})}{(z^{303}+x^{303})(z^{404}+x^{404})}$ {expanding summation}

$\Rightarrow$ $\frac{(1+1)(1+1)}{(1+1)(1+1)} + \frac{(1+1)(1+1)}{(1+1)(1+1)} + \frac{(1+1)(1+1)}{(1+1)(1+1)}$ {using (i)}

$\Rightarrow$ $1 + 1 + 1$

$\Rightarrow$ 3

Hence (C) is the correct answer.

Q12. Statement 1: for $x \geq 1$, $\sin^{-1}\left(\frac{2x}{1+x^2}\right) + \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \pi$.

Statement 2: for $x \geq 1$, $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \pi - 2\tan^{-1}x$ and $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1}x$.

(a) statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.

(b) statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.

(c) statement-1 is true, statement-2 is false.

(d) statement-1is false, statement-2 is ture.

Solution C

S_1 : we know that for $x \geq 1$,

$\text{Sin}^{-1}\left(\frac{2x}{1+x^2}\right) = \pi - 2\tan^{-1}x$ and $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \tan^{-1}x \Longrightarrow S_1$ is true. Also obviously S_2 is false.

Q13. Let function f(x) be defined as $f(x) = |\sin^{-1}x| + \cos^{-1}\left(\frac{1}{x}\right)$. Then which of the following is/are TRUE?

(a) f(x) is injective in its domain. (b) f(x) is many-one in its domain.

(c) Range of f is a singleton set. (d) None of these

Solution A

$f(x) = |\sin^{-1}x| + \cos^{-1}\left(\frac{1}{x}\right)$

Domain of f(x) is {-1, 1}

$F(1) = \frac{\pi}{2}$, $f(-1) = \frac{3\pi}{2}$

So function f (x) is injective

Sgn(f(x)) = 1 (f(x) > 0)

Range of f(x) $\left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$

Q14. f x = 2/3, then $\sin^2 (\tan^{-1} x) + \cos^2 (\sin^{-1} x)$ is equal to

A. $\frac{99}{101}$
B. $\frac{107}{117}$
C. $\frac{101}{117}$
D. None of these

Right Answer Explanation: C

$\tan^{-1} x = \sin^{-1} \frac{x}{\sqrt{1+x^2}}$, $\sin^{-1} x = \cos^{-1} \sqrt{1-x^2}$

So the required value is

$$\frac{x^2}{1+x^2}+1-x^2 = \frac{4/9}{1+4/9}+1-\frac{4}{9}=\frac{101}{117}.$$

Q15. The value of sin cot $^{-1}$ x is

(A) $\sqrt{1+x^2}$

(B) x

(C) $(1+x^2)^{-3/2}$

(D) $(1+x^2)^{-1/2}$

Solution D We have $\sin \cot^{-1} x = \sin \sin^{-1}[1/\sqrt{1+x^2}] = 1/\sqrt{1+x^2}$

Q16. The value of $\cos^{-1}(-1) - \sin^{-1}(1)$ is

(A) **π**

(B) $\frac{\pi}{2}$

(C) $\frac{3\pi}{2}$

(D) $-\frac{3\pi}{2}$

Solution B

$$E = \pi - \cos^{-1}(1) - \sin^{-1}(1) = \pi - \frac{\pi}{2} = \frac{\pi}{2}$$

$$\because \cos^{-1}(-x) = \pi - \cos^{-1} x$$

and $\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2}$

Q17 $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{2}{9} + \tan^{-1}\frac{4}{33} + \ldots\infty$ is equal to

(A) $\frac{\pi}{4}$

(B) π

(C) $\frac{\pi}{2}$

(D) none of these

Solution

The given series can be rewritten as

$$\tan^{-1}\frac{2^0}{1+2^1} + \tan^{-1}\frac{2^1}{1+2^3} + \tan^{-1}\frac{2^2}{1+2^5} + \ldots$$

$$\therefore T_n = \tan^{-1}\frac{2^{n-1}}{1+2^{2n-1}} = \tan^{-1}\frac{2^n - 2^{n-1}}{1+2^n.2^{n-1}}$$

or$T_n = \tan^{-1} 2^n - \tan^{-1} 2^{n-1}$

now put n = 1,2,3,..., and add. The terms cancel diagonally

$$\therefore S_n = \tan^{-1} 2^n - \tan^{-1}$$

$$\therefore S_\infty = \tan^{-1}(\infty) - \tan^{-1} 1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \rightarrow (a)$$

Q18. the equation $(\sin^{-1} x)^3 + (\cos^{-1} x)^3 = k\pi^3$ has no solution for

(A) $k > \frac{1}{32}$

(B) $k = \frac{1}{32}$

(C) $k < \frac{1}{32}$

(D) $k = 1$

Solution

$$(\sin^{-1} x + \cos^{-1} x)$$
$$\{(\sin^{-1} x)^2 + (\cos^{-1} x)^2 - \sin^{-1} x \cos^{-1} x\} = k\pi^3$$

$$\text{or}\ (\sin^{-1} x + \cos^{-1} x)^2 - 3\sin^{-1} x \cos^{-1} x = k\pi^3 . \frac{2}{\pi} = 2k\pi^2$$

$$\text{or}\ \frac{\pi^2}{4} - 3\sin^{-1} x \left(\frac{\pi}{2} - \sin^{-1} x\right) = 2k\pi^2$$

$$\text{or}\ 3(\sin^{-1} x)^2 - \frac{3\pi}{2}\sin^{-1} x + \pi^2\left(\frac{1}{4} - 2k\right) = 0$$

above is a quadratic equation and will have solution if

$$\Delta \geq 0\ or\ B^2 - 4AC \geq 0$$

$$\text{or}\ \frac{9\pi^2}{4} - 4.3.\pi^2\left(\frac{1}{4} - 2k\right) \geq 0$$

$$\text{or}\ 3 - 16\left(\frac{1}{4} - 2k\right) \geq 0\ or\ -1 + 32k \geq 0$$

or $k \geq \frac{1}{32}$. in this case it will have a solution.

Therefore, if $k < \frac{1}{32}$ it will have no solution.

Q19. The inequality $\sin^{-1} x > \cos^{-1} x$ holds for

(A) All values of x

(B) $x \in (0, 1/\sqrt{2})$

(C) $x \in (0, 1/\sqrt{2}, 1)$

(D) No value of x

Solution:

(c) Just take $\cos^{-1}$ as $\frac{\pi}{2} - \sin^{-1}x$ and Proceed

Q20. If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$, then $x^2 + y^2 + z^2 + 2xyz$ is equal to

(A) -1

(B) 0

(C) 1

(D) 5

Answer. C