IIT-JEE-Mathematics-Mains-2001

MAINS

Time : Three hours Max. Marks : 100

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1. Let a_1, a_2, …………. be positive real numbers in geometric progression. For each n, let A_n, G_n, H_n be respectively, the arithmetic mean, geometric mean, and harmonic mean of a_1, a_2, ………, a_n. Find an expression for the geometric mean of G_1, G_2, ………., G_n in terms of A_1, A_2, ………., A_n, H_1, H_2, ……….., H_n.

2. Let a, b, c be positive real numbers such that $b^2 - 4ac > 0$ and let $a_1 = c$. Prove

by induction that

$$a_{n+1} = (a\alpha_n^2) / ((b^2 - 2a (\alpha_1 - \alpha_2 + \ldots + \alpha_n))$$

is well-defined and $a_{n+1} < \alpha_n/2$ for all n = 1, 2, ………..

(Here, 'well-defined' means that the denominator in the expression for a_{n+1} is not zero.)

3. Let $-1 < p < 1$. Show that the equation $4x^3 - 3x - p = 0$ has a unique root in

the interval [1/2, 1] and identify it.

4. Let $2x^2 + y^2 - 3xy = 0$ be the equation of a pair of tangents drawn from the origin O to a circle of radius 3 with centre in the first quadrant. If A is one of the points of contact, find the length of OA.

5. Evaluate $\int \sin^{-1} ((2x+2) / \sqrt{4x^2+8x+13}) dx$.

6. Let f(x), $x \geq 0$, be a non-negative continuous function, and

let $F(x) = \int_0^x f(t)\, dt$, $x \geq 0$. If for some $c > 0$, $f(x) \leq cF(x)$ for all $x \geq 0$,

then show that f(x) = 0 for all x > 0.

7. Let $b \neq 0$ and for j = 0, 1, 2, …….., n, let S_j be the area of the region bounded

by the y-axis and the curve $xe^{ay} = \sin by$, $j\pi/b \leq y \leq ((j+1)\pi)/b$.

Show that S_0, S_1, S_2, ………., S_n are in geometric progression. Also, find their sum for a = – 1 and b = π.

8. Let $\alpha \in R$. Prove that a function f : R--> R is differentiable at α if and only if there is a function g : R --> R which is continuous at α and satisfies $f(x) - f(\alpha) = f(x)(x - \alpha)$ for all $x \in R$.

9. Let C_1 and C_2 be two circles with C_2 lying inside C_1. A circle C lying inside C_1 touches C_1 internally and C_2 externally. Identify the locus of the centre of C.

10. Show, by vector methods, that the angular bisectors of a triangle are concurrent and find an expression for the position vector of the point of concurrency in terms of the position vectors of the vertices.

11. (a) Let a, b, c be real numbers with a¹ 0 and let a, b be the roots of the equation ax2 + bx + c = 0. Express the roots of a2x2 + abcx + c3 = 0 in terms of a,b.

(b) Let a, b, c be real numbers with a2 + b2 + c2 = 1. Show that the equation

$$\begin{vmatrix} ax-by-c & bx+ay & cx+a \\ bx+ay & -ax+by-c & cy+b \\ cx+a & cy+b & -ax-by+c \end{vmatrix} = 0 \text{ represents a straight line.}$$

12. (a) Let P be a point on the ellipse x2/a2 + y2/b2 =1, 0 < b < a. Let the line parallel to y-axis passing through P meet the circle x2 + y2 = a2 at the point Q such that P and Q are on the same side of x-axis. For two positive real numbers r and s, find the locus of the point R on PQ such that PR : RQ = r : s as P varies over the ellipse.

(b) If D is the area of a triangle with side lengths a, b, c then

show that D < 1/4 √((a+b+c)abc).

Also show that the equality occurs in the above inequality if and only if a = b = c.

13. A hemispherical tank of radius 2 metres is initially full of water and has an outlet of 12 cm2 cross-sectional are at the bottom. The outlet is opened at some instant.

The flow through the outlet is according to the law v(t) = 0.6 √(2gh(t)), where v(t) and h(t) are respectively the velocity of the flow through the outlet and the height of water level above the outlet at time t, and g is the acceleration due to gravity. Find the time it takes to empty the tank.
(Hint : Form a differential equation by relating the decrease of water level to the outflow).

14. (a) An urn contains m white and n black balls. A ball is drawn at random and is put back into the urn along with k additional balls of the same colour as that of the ball drawn. A ball is again drawn at random. What is the probability that the ball drawn now is white?
(b) An unbiased die, with faces numbered 1, 2, 3, 4, 5, 6 is thrown n times and the list of n numbers showing up is noted. What is the probability that, among the numbers 1, 2, 3, 4, 5, 6 only three numbers appear in this list?

15. (a) Find 3-dimensional vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}$ satisfying
$\overrightarrow{v_1}.\overrightarrow{v_1} = 4, \overrightarrow{v_1}.\overrightarrow{v_2} = -2, \overrightarrow{v_1}.\overrightarrow{v_3} = 6, \overrightarrow{v_2}.\overrightarrow{v_2} = 2, \overrightarrow{v_2}.\overrightarrow{v_3} = -5, \overrightarrow{v_3}.\overrightarrow{v_3} = 29$.

(b) Let $\bar{A}(t) = f_1(t)\hat{\imath} + f_2(t)\hat{\jmath}$ and $B(t) = g_1(t)\hat{\imath} + g_2(t)\hat{\jmath}, t \in [0, 1]$, where f_1, f_2, g_1, g_2 are continuous functions. If $\bar{A}$ (t) and $\bar{B}$ (t) are non-zero vectors for all t and $\bar{A}(0) = 2\hat{\imath} + 3\hat{\jmath}, \bar{A}(1) = 6\hat{\imath} + 2\hat{\jmath}, \bar{B}(0) = 3\hat{\imath} + 2\hat{\jmath}$ and $\bar{B}(1) = 2\hat{\imath} + 6\hat{\jmath}$. The show that $\bar{A}$(t) and $\bar{B}$(t) are parallel for some t.