

# IIT-JEE-Mathematics-Mains-2003

## MAINS

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1. If  $z_1$  and  $z_2$  are two complex numbers such that

$|z_1| < 1 < |z_2|$  then prove that

$$\left| \frac{1 - z_1 \bar{z}_2}{z_1 - z_2} \right| < 1.$$

2. Find a point on the curve  $x^2 + 2y^2 = 6$  whose distance from the line  $x + y = 7$ , is minimum.

3. If matrix

$$A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$$

where  $a, b, c$  are real positive numbers,  $abc = 1$  and  $A^T A = I$ , then find the value of  $a^3 + b^3 + c^3$ .

4. Prove that

$$2^k \binom{n}{0} \binom{n}{k} - 2^{n-1} \binom{n}{1} \binom{n-1}{k-1} + 2^{k-2}$$

$$\binom{n}{2} \binom{n-2}{k-2} - \dots \dots \dots (-1)^k \binom{n}{k} \binom{n-k}{0} = \binom{n}{k}.$$

5. If  $f$  is an even function then prove that

$$\int_0^{\pi/2} f(\cos 2x) \cos x \, dx = \sqrt{2} \int_0^{\pi/4} f(\sin 2x) \cos x \, dx.$$

6. For a student to qualify, he must pass at least two out of three exams. The probability that he will pass the 1st exam is  $P$ . If he fails in one of the exams then the probability of his passing in the next exam is  $P/2$  otherwise it remains the same. Find the probability that he will qualify.

7. For the circle  $x^2 + y^2 = r^2$ , find the value of  $r$  for which the area enclosed by the tangents drawn from the point  $P(6, 8)$  to the circle and the chord of contact is maximum.

8. Prove that there exists no complex number  $z$  such that  $|z| < 1/3$  and

$$\prod_{r=1}^n \left( 1 + \frac{z}{r} \right) = 1 \quad \text{where } |z| < 2.$$

9. A is targeting to B, B and C are targeting to A. Probability of hitting the target by A, B and C are  $\frac{2}{3}, \frac{1}{2}$  and  $\frac{1}{3}$  respectively. If A is hit then find the probability that B hits the target and C does not.
10. If a function  $f : [-2a, 2a] \rightarrow \mathbb{R}$  is an odd function such that  $f(x) = f(2a - x)$  for  $x \in [a, 2a]$  and the left hand derivative at  $x = a$  is 0 then find the left hand derivative at  $x = -a$ .
11. Using the relation  $2(1 - \cos x) < x^2, x \neq 0$  or otherwise, prove that  $\sin(\tan x) > x \forall x \in [0, \frac{\pi}{4}]$ .
12. If  $a, b, c$  are in A.P.,  $a^2, b^2, c^2$  are in H.P., then prove that either  $a = b = c$  or  $a, b, -c/2$  form a G.P.
13. If  $x^2 + (a - b)x + (1 - a - b) = 0$  where  $a, b \in \mathbb{R}$  then find the values of  $a$  for which equation has unequal real roots for all values of  $b$ .
14. Normals are drawn from the point P with slopes  $m_1, m_2, m_3$  to the parabola  $y^2 = 4x$ . If locus of P with  $m_1 m_2 = a$  is a part of the parabola itself then find  $a$ .
15. If the function  $f : [0, 4] \rightarrow \mathbb{R}$  is differentiable then show that
- (i) For  $a, b \in (0, 4)$ ,  $(f(4))^2 = f(a) f(b)$
- (ii)  $\int_0^4 f(t) dt = 2[f(2) + f(4)] \forall 0 < a, b < 2$ .
16. (i) Find the equation of the plane passing through the points  $(2, 1, 0), (5, 0, 1)$  and  $(4, 1, 1)$
- (ii) If P is the point  $(2, 1, 6)$  then the point Q such that PQ is perpendicular to the plane in (i) and the mid point of PQ lies on it.
17. If  $P(1) = 0$  and  $(dP(x))/dx > P(x)$  for all  $x > 1$  then prove that  $P(x) > 0$  for all  $x > 1$ .
18. If  $I_n$  is the area of  $n$  sided regular polygon inscribed in a circle of unit radius and  $O_n$  be the area of the polygon circumscribing the given circle, prove that
- $$I_n = \frac{O_n}{2} \left( 1 + \sqrt{1 - \left(\frac{2I_n}{n}\right)^2} \right)$$
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19. If  $u, v, w$  are three non-coplanar unit vectors and  $\alpha, \beta, \gamma$  are the angles between  $u$  and  $v, v$  and  $w, w$  and  $u$  are respectively and  $x, y, z$  are unit vectors along the bisectors of the angles  $\alpha, \beta, \gamma$  respectively. Prove that

$$[\vec{x} \times \vec{y} \quad \vec{y} \times \vec{z} \quad \vec{z} \times \vec{x}] = \frac{1}{16} [\vec{u} \quad \vec{v} \quad \vec{w}]^2 \sec^2 \frac{\alpha}{2} \sec^2 \frac{\beta}{2} \sec^2 \frac{\gamma}{2}$$

**20.** A right circular cone with radius  $R$  and height  $H$  contains a liquid which evaporates at a rate proportional to its surface area in contact with air (proportionality constant =  $k > 0$ ). Find the time after which the cone is empty.