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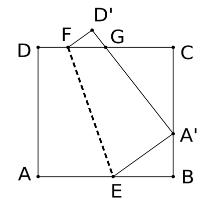
32nd Indian National Mathematical Olympiad-2017

Problems and Solutions

1. In the given figure, $ABCD$ is a square paper. It is folded along EF such that A goes to a point $A' \neq C, B$ on the side BC and D goes to D'. The line $A'D'$ cuts CD in G. Show that the inradius of the triangle GCA' is the sum of the inradii of the triangles $GD'F$ and $A'BE$.



Solution: Observe that the triangles GCA' and $A'BE$ are similar to the triangle $GD'F$. If $GF = u$, $GD' = v$ and $D'F = w$, then we have

$$A'G = pu, CG = pv, A'C = pw, \quad A'E = qu, BE = qw, A'B = qv.$$

If r is the inradius of $\triangle GD'F$, then pr and qr are respectively the inradii of triangles GCA' and $A'BE$. We have to show that $pr = r + qr$. We also observe that

$$AE = EA', \quad DF = FD'.$$

Therefore

$$pw + qv = qw + qu = w + u + pv = v + pu.$$

The last two equalities give $(p-1)(u-v) = w$. The first two equalities give $(p-q)w = q(u-v)$. Hence

$$\frac{p-q}{q} = \frac{u-v}{w} = \frac{1}{p-1}.$$

This simplifies to $p(p-q-1) = 0$. Since $p \neq 0$, we get $p = q+1$. This implies that $pr = qr + r$.

2. Suppose $n \geq 0$ is an integer and all the roots of $x^3 + \alpha x + 4 - (2 \times 2016^n) = 0$ are integers. Find all possible values of α.

Solution 1: Let u, v, w be the roots of $x^3 + \alpha x + 4 - (2 \times 2016^n) = 0$. Then $u + v + w = 0$ and $uvw = -4 + (2 \times 2016^n)$. Therefore we obtain

$$uv(u + v) = 4 - (2 \times 2016^n).$$

Suppose $n \geq 1$. Then we see that $uv(u + v) \equiv 4 \pmod{2016^n}$. Therefore $uv(u + v) \equiv 1 \pmod 3$ and $uv(u + v) \equiv 1 \pmod 9$. This implies that $u \equiv 2 \pmod 3$ and $v \equiv 2 \pmod 3$. This shows that modulo 9 the pair (u, v) could be any one of the following:

$$(2, 2), (2, 5), (2, 8), (5, 2), (5, 5), (5, 8), (8, 2), (8, 5), (8, 8).$$

In each case it is easy to check that $uv(u + v) \not\equiv 4 \pmod 9$. Hence $n = 0$ and $uv(u + v) = 2$. It follows that $(u, v) = (1, 1)$, $(1, -2)$ or $(-2, 1)$. Thus

$$\alpha = uv + vw + wu = uv - (u + v)^2 = -3$$

for every pair (u, v).

Solution 2: Let $a, b, c \in \mathbb{Z}$ be the roots of the given equation for some $n \in \mathbb{N}_0$. By Vieta Theorem, we know that

$$a + b + c = 0$$

$$ab + bc + ca = \alpha$$

$$abc = 2 \times 2016^n - 4$$

If possible, let us have $n \geq 1$. Since $7|2016$, we have that

$$7|abc + 4 \implies 7|3(abc + 4) \implies 7|3abc + 12 \implies 7|3abc + 5$$

Since we have $a + b + c = 0$, we get that $3abc = a^3 + b^3 + c^3$. Substituting this in the earlier expression, we get that

$$a^3 + b^3 + c^3 + 5 \equiv 0 \pmod 7$$

Consider below, a table calculating the residues of cubes modulo 7.

x	0	1	2	3	4	5	6
x^3	0	1	1	-1	1	-1	-1

Hence, we know that if $x \in \mathbb{N}$, then we have $x^3 \equiv 0, 1, -1 \pmod 7$. Since $a^3 + b^3 + c^3 \equiv 2 \pmod 7$, we see that we must have one of the numbers divisible by 7 and the other two numbers, when cubed, must leave 1 as remainder modulo 7. Without of generality, let us assume that

$$a \equiv 0 \pmod 7, \quad b^3, c^3 \equiv 1 \pmod 7$$

Hence, we have $b, c \equiv 1, 2, 4 \pmod 7$. We will consider all possible values of $b + c$ modulo 7. Since the expression is symmetric in b, c, modulo 7, we will consider $b \leq c$.

b	1	1	1	2	2	4
c	1	2	4	2	4	4
$b + c$	2	3	5	4	6	1

We see that, in all the above cases, we get $7 \nmid b+c$. But this is a contradiction, since $7|a+b+c$ and $7|a$ together imply that $7|b + c$. Hence, we cannot have $n \geq 1$. Hence, the only possible value is $n = 0$. Substituting this value in the original equation, the equation becomes

$$x^3 + \alpha x + 2 = 0$$

Solving the equations $a + b + c = 0$ and $abc = -2$ in integers, we see that the only possible solutions (a, b, c) are permutations of $(1, 1, -2)$. In case of any permutation, $\alpha = -3$. Substituting this value of α back in the equation, we see that we indeed, get integer roots. Hence, the only possible value for α is -3.

3. Find the number of triples (x, a, b) where x is a real number and a, b belong to the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that

$$x^2 - a\{x\} + b = 0,$$

where $\{x\}$ denotes the fractional part of the real number x. (For example $\{1.1\} = 0.1 = \{-0.9\}$.)

Solution: Let us write $x = n + f$ where $n = [x]$ and $f = \{x\}$. Then

$$f^2 + (2n - a)f + n^2 + b = 0. \tag{1}$$

Observe that the product of the roots of (1) is $n^2 + b \geq 1$. If this equation has to have a solution $0 \leq f < 1$, the larger root of (1) is greater 1. We conclude that the equation (1) has a real root less than 1 only if $P(1) < 0$ where $P(y) = y^2 + (2n - a)y + n^2 + 2b$. This gives

$$1 + 2n - a + n^2 + 2b < 0.$$

Therefore we have $(n + 1)^2 + b < a$. If $n \geq 2$, then $(n + 1)^2 + b \geq 10 > a$. Hence $n \leq 1$. If $n \leq -4$, then again $(n + 1)^2 + b \geq 10 > a$. Thus we have the range for n: $-3, -2, -1, 0, 1$.

If $n = -3$ or $n = 1$, we have $(n + 1)^2 = 4$. Thus we must have $4 + b < a$. If $a = 9$, we must have $b = 4, 3, 2, 1$ giving 4 values. For $a = 8$, we must have $b = 3, 2, 1$ giving 3 values. Similarly, for $a = 7$ we get 2 values of b and $a = 6$ leads to 1 value of b. In each case we get a real value of $f < 1$ and this leads to a solution for x. Thus we get totally $2(4+3+2+1) = 20$ values of the triple (x, a, b).

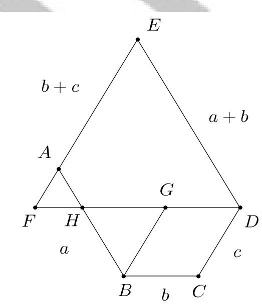
For $n = -2$ and $n = 0$, we have $(n + 1)^2 = 1$. Hence we require $1 + b < a$. We again count pairs (a, b) such that $a - b > 1$. For $a = 9$, we get 7 values of b; for $a = 8$ we get 6 values of b and so on. Thus we get $2(7 + 6 + 5 + 4 + 3 + 2 + 1) = 56$ values for the triple (x, a, b).

Suppose $n = -1$ so that $(n + 1)^2 = 0$. In this case we require $b < a$. We get $8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 36$ values for the triple (x, a, b).

Thus the total number of triples (x, a, b) is $20 + 56 + 36 = 112$.

4. Let $ABCDE$ be a convex pentagon in which $\angle A = \angle B = \angle C = \angle D = 120°$ and whose side lengths are 5 consecutive integers in some order. Find all possible values of $AB + BC + CD$.

Solution 1: Let $AB = a$, $BC = b$, and $CD = c$. By symmetry, we may assume that $c < a$. We show that $DE = a + b$ and $EA = b + c$.



Draw a line parallel to BC through D. Extend EA to meet this line at F. Draw a line parallel to CD through B and let it intersect DF in G. Let AB intersect DF in H. We have $\angle FDE = 60°$ and $\angle E = 60°$. Hence EFD is an equilateral triangle. Similarly AFH and BGH are also equilateral triangles. Hence $HG = GB = c$. Moreover, $DG = b$. Therefore $HD = b + c$. But $HD = AE$ since $FH = FA$ and $FD = FE$. Also $AH = a - BH = a - BG = a - c$. Hence $ED = EF = EA + AF = b + c + AH = (b + c) + (a - c) = b + a$.

We have five possibilities:
(1) $b < c < a < b + c < a + b$;

(2) $c < b < a < b+c < a+b$;
(3) $c < a < b < b+c < a+b$;
(4) $b < c < b+c < a < a+b$;
(5) $c < b < b+c < a < a+b$.

In (1), we see that $c < a < b+c$ are three consecutive integers provided $b = 2$. Hence we get $c = 3$ and $a = 4$. In this case $b+c = 5$ and $a+b = 6$ so that we have five consecutive integrs $2, 3, 4, 5, 6$ as side lengths. In (2), $b < a < b+c$ form three consecutive integrs only when $c = 2$. Hence $b = 3$, $a = 4$. But then $b+c = 5$ and $a+b = 7$. Thus the side lengths are $2, 3, 4, 6, 7$ which are not consecutive integers. In case (3), $b < b+c$ are two consecutive integrs so that $c = 1$. Hence $a = 2$ and $b = 3$. We get $b+c = 4$ and $a+b = 5$ so that the consecutive integers $1, 2, 3, 4, 5$ form the side lengths. In case (4), we have $c < b+c$ as two consecutive integers and hence $b = 1$. Therefore $c = 2$, $b+c = 3$, $a = 4$ and $a+b = 5$ which is admissible. Finally, in case (5) we have $b < b+c$ as two consecutive integers, so that $c = 1$. Thus $b = 2$, $b+c = 3$, $a = 4$ and $a+b = 6$. We do not get consecutive integers.

Therefore the only possibilities are $(a, b, c) = (4, 2, 3)$, $(2, 3, 1)$ and $(4, 1, 2)$. This shows that $a+b+c = 9$, 6 or 7. Thus there are three possible sums $AB + BC + CA$, namely, 6, 7 or 9.

Solution 2: As in the earlier solution, $ED = d = a+b$ and $EA = e = b+c$. Let the sides be $x-2, x-1, x, x+1, x+2$. Then $x \geq 3$. We also have $x+2 \geq x-1+x-2$ so that $x \leq 5$. Thus $x = 3, 4$ or 5. If $x = 5$, the sides are $\{3, 4, 5, 6, 7\}$ and here we do not have two pairs which add to a number in the set. Hence $x = 3$ or 4 and we get the sets as $\{1, 2, 3, 4, 5\}$ or $\{2, 3, 4, 5, 6\}$. With the set $\{1, 2, 3, 4, 5\}$ we get

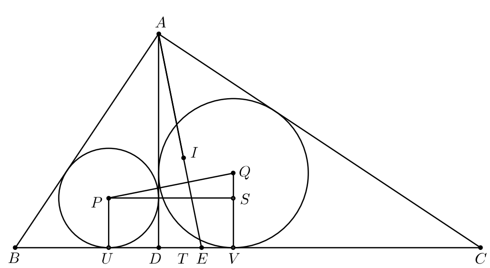
$$(a, b, c, d, e) = (2, 3, 1, 5, 4), (4, 1, 2, 5, 3).$$

From the set $\{2, 3, 4, 5, 6\}$, we get $(a, b, c, d, e) = (4, 2, 3, 6, 5)$. Thus we see that $a+b+c = 6, 7$ or 9.

Solution 3: We use the same notations and we get $d = a+b$ and $e = b+c$. If $a \geq 5$, we see that $d - b \geq 5$. But the maximum difference in a set of 5 consecutive integers is 4. Hence $a \leq 4$. Similarly, we see $b \leq 4$ and $c \leq 4$. Thus we see that $a+b+c \leq 2+3+4 = 9$. But $a+b+c \geq 1+2+3 = 6$. It follows that $a+b+c = 6, 7, 8$ or 9. If we take $(a, b, c, d, e) = (1, 3, 2, 4, 5)$, we get $a+b+c = 6$. Similarly, $(a, b, c, d, e) = (2, 1, 4, 3, 5)$ gives $a+b+c = 7$, For $a+b+c = 8$, the only we we can get $1+3+4 = 8$. Here we cannot accommodate 2 and consecutiveness is lost. For 9, we can have $(a, b, c, d, e) = (3, 2, 4, 5, 6)$ and $a+b+c = 9$.

5. Let ABC be a triangle with $\angle A = 90°$ and $AB < AC$. Let AD be the altitude from A on to BC. Let P, Q and I denote respectively the incentres of triangles ABD, ACD and ABC. Prove that AI is perpendicular to PQ and $AI = PQ$.

Solution: Draw $PS \parallel BC$ and $QS \parallel AD$. Then PSQ is a right-angled triangle with $\angle PSQ = 90°$. Observe that $PS = r_1 + r_2$ and $SQ = r_2 - r_1$, where r_1 and r_2 are the inradii of triangles ABD and ACD, respectively. We observe that triangles DAB and DCA are similar to triangle ACB.



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Hence

$$r_1 = \frac{c}{a}r, \quad r_2 = \frac{b}{a}r,$$

where r is the inradius of triangle ABC. Thus we get

$$\frac{PS}{SQ} = \frac{r_2 + r_1}{r_2 - r_1} = \frac{b+c}{b-c}.$$

On the otherhand $AD = h = bc/a$. We also have $BE = ca/(b+c)$ and

$$BD^2 = c^2 - h^2 = c^2 - \frac{b^2c^2}{a^2} = \frac{c^4}{a^2}.$$

Hence $BD = c^2/a$. Therefore

$$DE = BE - BD = \frac{ca}{b+c} - \frac{c^2}{a} = \frac{cb(b-c)}{a(b+c)}.$$

Thus we get

$$\frac{AD}{DE} = \frac{b+c}{b-c} = \frac{PS}{SQ}.$$

Since $\angle ADE = 90^\circ = \angle PSQ$, we conclude that $\triangle ADE \sim \triangle PSQ$. Since $AD \perp PS$, it follows that $AE \perp PQ$.

We also observe that

$$PQ^2 = PS^2 + SQ^2 = (r_2 + r_1)^2 + (r_2 - r_1)^2 = 2(r_1^2 + r_2^2).$$

However

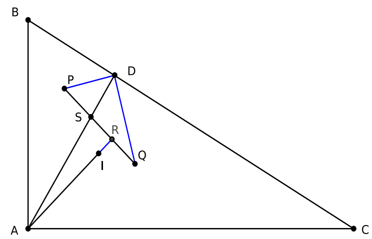
$$r_1^2 + r_2^2 = \frac{c^2 + b^2}{a^2}r^2 = r^2.$$

Hence $PQ = \sqrt{2}r$. We also observe that $AI = r\text{cosec}(A/2) = r\text{cosec}(45^\circ) = \sqrt{2}r$. Thus $PQ = AI$.

Solution 2: In the figure, we have made the construction as mentioned in the hint. Since P, Q are the incentres of $\triangle ABD, \triangle ACD$, DP, DQ are the internal angle bisectors of $\angle ADB, \angle ADC$ respectively. Since AD is the altitude on the hypotenuse BC in $\triangle ABC$, we have that $\angle PDQ = 45^\circ + 45^\circ = 90^\circ$. It also implies that

$$\triangle ABC \sim \triangle DBA \sim \triangle DAC$$

This implies that all corresponding length in the above mentioned triangles have the same ratio.



In particular,

$$\begin{aligned}
& \frac{AI}{BC} = \frac{DP}{AB} = \frac{DQ}{AC} \\
\implies & \frac{AI^2}{BC^2} = \frac{DP^2}{AB^2} = \frac{DQ^2}{AC^2} = \frac{DP^2 + DQ^2}{AB^2 + AC^2} \\
\implies & \frac{AI^2}{BC^2} = \frac{PQ^2}{BC^2}, \quad \text{by Pythagoras Theorem in } \triangle ABC, \triangle PDQ \\
\implies & AI = PQ
\end{aligned}$$

as required.

For the second, part, we note that from the above relations, we have $\triangle ABC \sim \triangle DPQ$. Let us take $\angle ACB = \theta$. Then, we get

$$\begin{aligned}
\angle PSD &= 180^\circ - (\angle SPD + \angle SDP) \\
&= 180^\circ - (90^\circ - \theta + 45^\circ) \\
&= 45^\circ + \theta
\end{aligned}$$

This gives us that

$$\begin{aligned}
\angle ARS &= 180^\circ - (\angle ASR + \angle SAR) \\
&= 180^\circ - (\angle PSD + \angle SAC - \angle IAC) \\
&= 180^\circ - (45^\circ + \theta + 90^\circ - \theta - 45^\circ) \\
&= 90^\circ
\end{aligned}$$

as required. Hence, we get that $AI = PQ$ and $AI \perp PQ$.

Solution 3: We know that the angle bisector of $\angle B$ passes through P, I which implies that B, P, I are collinear. Similarly, C, Q, I are also collinear. Since I is the incentre of $\triangle ABC$, we know that

$$\angle PIQ = \angle BIC = 90^\circ + \frac{\angle A}{2} = 135^\circ$$

Join AP, AQ. We know that $\angle BAP = \frac{1}{2}\angle BAD = \frac{1}{2}\angle C$. Also, $\angle ABP = \frac{1}{2}\angle B$. Hence by Exterior Angle Theorem in $\triangle ABP$, we get that

$$\angle API = \angle ABP + \angle BAP = \frac{1}{2}(\angle B + \angle C) = 45^\circ$$

Similarly in $\triangle ADC$, we get that $\angle AQI = 45^\circ$. Also, we have

$$\angle PAI = \angle BAI - \angle BAP = 45^\circ - \frac{\angle C}{2} = \frac{\angle B}{2}$$

Similarly, we get $\angle QAI = \frac{\angle C}{2}$.

Now applying Sine Rule in $\triangle API$, we get

$$\frac{IP}{\sin\angle PAI} = \frac{AI}{\sin\angle API} \implies IP = \sqrt{2}AI\sin\frac{B}{2}$$

Similarly, applying Sine Rule in $\triangle AQI$, we get

$$\frac{IQ}{\sin\angle PAI} = \frac{AI}{\sin\angle AQI} \implies IQ = \sqrt{2}AI\sin\frac{C}{2}$$

Applying Cosine Rule in $\triangle PIQ$ gives us that

$$\begin{aligned} PQ^2 &= IP^2 + IQ^2 - 2\cdot IP\cdot IQ\cos\angle PIQ \\ &= 2AI^2\left(\sin^2\frac{B}{2} + \sin^2\frac{C}{2} + \sqrt{2}\sin\frac{B}{2}\sin\frac{C}{2}\right) \end{aligned}$$

We will prove that $\left(\sin^2\frac{B}{2} + \sin^2\frac{C}{2} + \sqrt{2}\sin\frac{B}{2}\sin\frac{C}{2}\right) = \frac{1}{2}$. In any $\triangle XYZ$, we have that

$$\sum_{cyc}\sin^2\frac{X}{2} = 1 - 2\prod\sin\frac{X}{2}$$

Using this in $\triangle ABC$, and using the fact that $\angle A = 90^\circ$, we get

$$\begin{aligned} & \sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} = 1 - 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \\ \implies & \frac{1}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} = 1 - \sqrt{2}\sin\frac{B}{2}\sin\frac{C}{2} \\ \implies & \left(\sin^2\frac{B}{2} + \sin^2\frac{C}{2} + \sqrt{2}\sin\frac{B}{2}\sin\frac{C}{2}\right) = \frac{1}{2} \end{aligned}$$

which was to be proved. Hence we get $PQ = AI$.

The second part of the problem can be obtained by angle-chasing as outlined in Solution 2.

Solution 4: Observe that $\angle APB = \angle AQC = 135^\circ$. Thus $\angle API = \angle AQI = 45^\circ$ (since $B - P - I$ and $C - Q - I$). Note $\angle PAQ = 1/2\angle A = 45^\circ$. Let $X = BI \cap AQ$ and $Y = CI \cap AP$. Therefore $\angle AXP = 180 - \angle API - \angle PAQ = 90^\circ$. Similarly $\angle AYQ = 90^\circ$. Hence I is the orthocentre of triangle PAQ. Therefore AI is perpendicular to PQ. Also $AI = 2R_{PAQ}\cos 45^\circ = 2R_{PAQ}\sin 45^\circ = PQ$.

6. Let $n \geq 1$ be an integer and consider the sum

$$x = \sum_{k\geq 0} \binom{n}{2k} 2^{n-2k} 3^k = \binom{n}{0} 2^n + \binom{n}{2} 2^{n-2} \cdot 3 + \binom{n}{4} 2^{n-4} \cdot 3^2 + \cdots \quad .$$

Show that $2x-1, 2x, 2x+1$ form the sides of a triangle whose area and inradius are also integers.

Solution: Consider the binomial expansion of $(2+\sqrt{3})^n$. It is easy to check that

$$(2+\sqrt{3})^n = x + y\sqrt{3},$$

where y is also an integer. We also have

$$(2-\sqrt{3})^n = x - y\sqrt{3}.$$

Multiplying these two relations, we obtain $x^2 - 3y^2 = 1$.
Since all the terms of the expansion of $(2+\sqrt{3})^n$ are positive, we see that

$$2x = (2+\sqrt{3})^n + (2-\sqrt{3})^n = 2\left(2^n + \binom{n}{2} 2^{n-2} \cdot 3 + \cdots\right) \geq 4.$$

Thus $x \geq 2$. Hence $2x+1 < 2x + (2x-1)$ and therefore $2x-1, 2x, 2x+1$ are the sides of a triangle. By Heron's formula we have

$$\Delta^2 = 3x(x+1)(x)(x-1) = 3x^2(x^2-1) = 9x^2y^2.$$

Hence $\Delta = 3xy$ which is an integer. Finally, its inradius is

$$\frac{\text{area}}{\text{perimeter}} = \frac{3xy}{3x} = y,$$

which is also an integer.

Solution 2: We will first show that the numbers $2x_n - 1, 2x_n, 2x_n + 1$ form the sides of a triangle. To show that, it suffices to prove that $2x_n - 1 + 2x_n > 2x_n + 1$. If possible, let the converse hold. Then, we see that we must have $4x_n - 1 \leq 2x_n + 1$, which implies that $x_n \leq 1$. But we see that even for the smallest value of $n = 1$, we have that $x_n > 1$. Hence, the numbers are indeed sides of a triangle.

Let Δ_n, r_n, s_n denote respectively, the area, inradius and semiperimeter of the triangle with sides $2x_n - 1, 2x_n, 2x_n + 1$. By Heron's Formula for the area of a triangle, we see that

$$\Delta_n = \sqrt{3x_n(x_n-1)x_n(x_n+1)} = x_n\sqrt{3(x_n^2-1)}$$

If possible, let Δ_n be an integer for all $n \in \mathbb{N}$. We see that due to the presence of the first term $\binom{n}{0}2^n$, we have $3 \nmid x_n, \ \forall n \in \mathbb{N}$. Hence, we get that $3 | x_n^2 - 1$. Hence, we can write $x_n^2 - 1$ as $3m$ for some $m \in \mathbb{N}$. Then, we can also write

$$\Delta_n = 3x_n\sqrt{m}$$

Note that we have assumed that Δ_n is an integer. Hence, we see that we must have m to be a perfect square. Consequently, we get that

$$r_n = \frac{\Delta_n}{s_n} = \frac{\Delta_n}{3x_n} = \sqrt{m} \in \mathbb{Z}$$

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Hence, it only remains to show that $\Delta_n \in \mathbb{Z},\ \forall n \in \mathbb{N}$. In other words, it suffices to show that $3(x_n^2 - 1)$ is a perfect square for all $n \in \mathbb{N}$.

We see that we can write x_n as

$$\begin{aligned}
x_n &= \frac{1}{2}\left(2\sum_{k\geq 0}\binom{n}{2k}2^{n-2k}3^k\right) \\
&= \frac{1}{2}\left((2+\sqrt{3})^n + (2-\sqrt{3})^n\right) \\
3x_n^2 - 3 &= \frac{3}{4}\left((2+\sqrt{3})^{2n} + (2-\sqrt{3})^{2n} + 2(2+\sqrt{3})^n(2-\sqrt{3})^n\right) - 3 \\
&= \frac{3}{4}\left((2+\sqrt{3})^{2n} + (2-\sqrt{3})^{2n} - 2(2+\sqrt{3})^n(2-\sqrt{3})^n\right) \\
&= \left(\frac{\sqrt{3}}{2}\left((2+\sqrt{3})^n - (2-\sqrt{3})^n\right)\right)^2
\end{aligned}$$

We are left to show that the quantity obtained in the above equation is an integer. But we see that if we define

$$a_n = \frac{\sqrt{3}}{2}\left((2+\sqrt{3})^n - (2-\sqrt{3})^n\right),\ \ \forall n \in \mathbb{N}$$

the sequence $\langle a_k\rangle_{k=1}^{\infty}$ thus obtained is exactly the solution for the recursion given by

$$a_{n+2} = 4a_{n+1} - a_n,\ \ \forall n \in \mathbb{N},\ \ a_1 = 3, a_2 = 12$$

Hence, clearly, each a_n is obviously an integer, thus completing the proof.

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