## CRMO-2015 questions and solutions

1. In a cyclic quadrilateral $A B C D$, let the diagonals $A C$ and $B D$ intersect at $X$. Let the circumcircles of triangles $A X D$ and $B X C$ intersect again at $Y$. If $X$ is the incentre of triangle $A B Y$, show that $\angle C A D=90^{\circ}$.

Solution: Given that $X$ is the incentre of triangle $A B Y$, we have $\angle B A X=\angle X A Y$. Therefore, $\angle B D C=\angle B A C=\angle B A X=\angle X A Y=$ $\angle X D Y=\angle B D Y$. This shows that $C, D, Y$ are collinear. Therefore, $\angle C Y X+\angle X Y D=180^{\circ}$. But the left-hand side equals $\left(180^{\circ}-\angle C B D\right)+$ $\left(180^{\circ}-\angle C A D\right)$. Since $\angle C B D=\angle C A D$, we ob-
 tain $180^{\circ}=360^{\circ}-2 \angle C A D$. This shows that $\angle C A D=90^{\circ}$.
2. Let $P_{1}(x)=x^{2}+a_{1} x+b_{1}$ and $P_{2}(x)=x^{2}+a_{2} x+b_{2}$ be two quadratic polynomials with integer coefficients. Suppose $a_{1} \neq a_{2}$ and there exist integers $m \neq n$ such that $P_{1}(m)=P_{2}(n)$, $P_{2}(m)=P_{1}(n)$. Prove that $a_{1}-a_{2}$ is even.

Solution: We have

$$
\begin{aligned}
m^{2}+a_{1} m+b_{1} & =n^{2}+a_{2} n+b_{2} \\
n^{2}+a_{1} n+b_{1} & =m^{2}+a_{2} m+b_{2}
\end{aligned}
$$

Hence

$$
\left(a_{1}-a_{2}\right)(m+n)=2\left(b_{2}-b_{1}\right), \quad\left(a_{1}+a_{2}\right)(m-n)=2\left(n^{2}-m^{2}\right)
$$

This shows that $a_{1}+a_{2}=-2(n+m)$. Hence

$$
4\left(b_{2}-b_{1}\right)=a_{2}^{2}-a_{1}^{2}
$$

Since $a_{1}+a_{2}$ and $a_{1}-a_{2}$ have same parity, it follows that $a_{1}-a_{2}$ is even.
3. Find all fractions which can be written simultaneously in the forms $\frac{7 k-5}{5 k-3}$ and $\frac{6 l-1}{4 l-3}$, for some integers $k, l$.

Solution: If a fraction is simultaneously in the forms $\frac{7 k-5}{5 k-3}$ and $\frac{6 l-1}{4 l-3}$, we must have

$$
\frac{7 k-5}{5 k-3}=\frac{6 l-1}{4 l-3}
$$

This simplifies to $k l+8 k+l-6=0$. We can write this in the form

$$
(k+1)(l+8)=14
$$

Now 14 can be factored in 8 ways: $1 \times 14,2 \times 7,7 \times 2,14 \times 1,(-1) \times(-14),(-2) \times(-7)$, $(-7) \times(-2)$ and $(-14) \times(-1)$. Thus we get 8 pairs:

$$
(k, l)=(13,-7),(6,-6),(1,-1),(0,6),(-15,-9),(-8,-10),(-3,-15),(-2,-22)
$$

These lead respectively to 8 fractions:

$$
\frac{43}{31}, \quad \frac{31}{27}, \quad 1, \quad \frac{55}{39}, \quad \frac{5}{3}, \quad \frac{61}{43}, \quad \frac{19}{13}, \quad \frac{13}{9} .
$$

4. Suppose 28 objects are placed along a circle at equal distances. In how many ways can 3 objects be chosen from among them so that no two of the three chosen objects are adjacent nor diametrically opposite?

Solution: One can choose 3 objects out of 28 objects in $\binom{28}{3}$ ways. Among these choices all would be together in 28 cases; exactly two will be together in $28 \times 24$ cases. Thus three objects can be chosen such that no two adjacent in $\binom{28}{3}-28-(28 \times 24)$ ways. Among these, furthrer, two objects will be diametrically opposite in 14 ways and the third would be on either semicircle in a non adjacent portion in $28-6=22$ ways. Thus required number is

$$
\binom{28}{3}-28-(28 \times 24)-(14 \times 22)=2268
$$

5. Let $A B C$ be a right triangle with $\angle B=90^{\circ}$. Let $E$ and $F$ be respectively the mid-points of $A B$ and $A C$. Suppose the incentre $I$ of triangle $A B C$ lies on the circumcircle of triangle $A E F$. Find the ratio $B C / A B$.

Solution: Draw $I D \perp A C$. Then $I D=r$, the inradius of $\triangle A B C$. Observe $E F \| B C$ and hence $\angle A E F=\angle A B C=90^{\circ}$. Hence $\angle A I F=90^{\circ}$. Therefore $I D^{2}=F D \cdot D A$. If $a>c$, then $F A>D A$ and we have

$$
D A=s-a, \quad \text { and } \quad F D=F A-D A=\frac{b}{2}-(s-a)
$$

Thus we obtain

$$
r^{2}=\frac{(b+c-a)(a-c)}{4}
$$

But $r=(c+a-b) / 2$. Thus we obtain

$$
(c+a-b)^{2}=(b+c-a)(a-c)
$$

Simplification gives $3 b=3 a+c$. Squaring both sides and using $b^{2}=c^{2}+a^{2}$, we obtian $4 c=3 a$. Hence $B C / B A=a / c=4 / 3$.
(If $a \leq c$, then $I$ lies outside the circumcircle of $A E F$.)
6. Find all real numbers $a$ such that $3<a<4$ and $a(a-3\{a\})$ is an integer. (Here $\{a\}$ denotes the fractional part of $a$. For example $\{1.5\}=0.5 ;\{-3.4\}=0.6$.)

Solution: Let $a=3+f$, where $0<f<1$. We are given that $(3+f)(3-2 f)$ is an integer. This implies that $2 f^{2}+3 f$ is an integer. Since $0<f<1$, we have $0<2 f^{2}+3 f<5$. Therefore $2 f^{2}+3 f$ can take $1,2,3$ or 4 . Equating $2 f^{2}+3 f$ to each one of them and using $f>0$, we get

$$
f=\frac{-3+\sqrt{17}}{4}, \quad \frac{1}{2}, \quad \frac{-3+\sqrt{33}}{4}, \quad \frac{-3+\sqrt{41}}{4}
$$

Therefore $a$ takes the values:

$$
a=3+\frac{-3+\sqrt{17}}{4}, \quad 3 \frac{1}{2}, \quad 3+\frac{-3+\sqrt{33}}{4}, \quad 3+\frac{-3+\sqrt{41}}{4} .
$$

