# Regional Mathematical Olympiad-2017
## Solutions

1. Let $AOB$ be a given angle less than $180°$ and let $P$ be an interior point of the angular region determined by $\angle AOB$. Show, with proof, how to construct, using only ruler and compasses, a line segment $CD$ passing through $P$ such that $C$ lies on the ray $OA$ and $D$ lies on the ray $OB$, and $CP : PD = 1 : 2$.

   **Solution:** Draw a line parallel to $OA$ through $P$. Let it intersect $OB$ in $M$. Using compasses, draw an arc of a circle with centre $M$ and radius $MO$ to cut $OB$ in $L$, $L \neq O$. Again with $L$ as centre and with the same radius $OM$ draw one more arc of a circle to cut $OB$ in $D$, $D \neq M$. Join $DP$ and extend it to meet $OA$ in $C$. Then $CD$ is the required line segment such that $CP : PD = 1 : 2$. This follows from similar triangles $OCD$ and $MPD$.
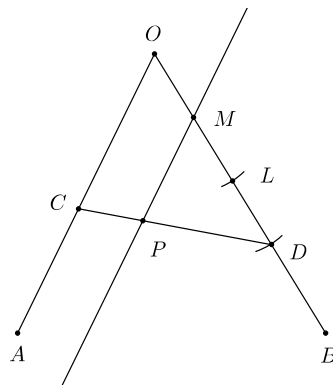


2. Show that the equation

   $$a^3 + (a+1)^3 + (a+2)^3 + (a+3)^3 + (a+4)^3 + (a+5)^3 + (a+6)^3 = b^4 + (b+1)^4$$

   has no solutions in integers $a$, $b$.

   **Solution:** We use divisibility argument by $7$. Observe that the remainders of seven consecutive cubes modulo $7$ are $0, 1, 1, 6, 1, 6, 6$ in some (cyclic) order. Hence the sum of seven consecutive cubes is $0$ modulo $7$. On the other hand the remainders of two consecutive fourth powers modulo $7$ is one of the sets $\{0, 1\}$, $\{1, 2\}$, $\{2, 4\}$, $\{4, 4\}$. Hence the sum of two fourth powers is never divisible by $7$. It follows that the given equation has no solution in integers.

3. Let $P(x) = x^2 + \frac{1}{2}x + b$ and $Q(x) = x^2 + cx + d$ be two polynomials with real coefficients such that $P(x)Q(x) = Q(P(x))$ for all real $x$. Find all the real roots of $P(Q(x)) = 0$.

**Solution:** Observe that

$$P(x)Q(x) = x^4 + \left(c + \frac{1}{2}\right)x^3 + \left(b + \frac{c}{2} + d\right)x^2 + \left(\frac{d}{2} + bc\right)x + bd.$$

Similarly,

$$
\begin{aligned}
Q(P(x)) &= \left(x^2 + \frac{1}{2}x + b\right)^2 + c\left(x^2 + \frac{1}{2}x + b\right) + d \\
&= x^4 + x^3 + \left(2b + \frac{1}{4} + c\right)x^2 + \left(b + \frac{c}{2}\right)x + b^2 + bc + d.
\end{aligned}
$$

Equating coefficients of corresponding powers of $x$, we obtain

$$c + \frac{1}{2} = 1, \quad b + \frac{c}{2} + d = 2b + \frac{1}{4} + c, \quad \frac{d}{2} + bc = b + \frac{c}{2}, \quad b^2 + bc + d = bd.$$

Solving these, we obtain

$$c = \frac{1}{2}, d = 0, b = \frac{-1}{2}.$$

Thus the polynomials are

$$P(x) = x^2 + \frac{1}{2}x - \frac{1}{2}, \quad Q(x) = x^2 + \frac{1}{2}x.$$

Therefore,

$$
\begin{aligned}
P(Q(x)) &= \left(x^2 + \frac{1}{2}x\right)^2 + \frac{1}{2}\left(x^2 + \frac{1}{2}x\right) - \frac{1}{2} \\
&= x^4 + x^3 + \frac{3}{4}x^2 + \frac{1}{4}x - \frac{1}{2}.
\end{aligned}
$$

It is easy to see that

$$P(Q(-1)) = 0, \quad P(Q(1/2)) = 0.$$

Thus $(x + 1)$ and $(x - 1/2)$ are factors of $P(Q(x))$. The remaining factor is

$$h(x) = x^2 + \frac{1}{2}x + 1.$$

The discriminant of $h(x)$ is $D = (1/4) - 4 < 0$. Hence $h(x) = 0$ has no real roots. Therefore the only real roots of $P(Q(x)) = 0$ are $-1$ and $1/2$.

4. Consider $n^2$ unit squares in the $xy$-plane centred at point $(i, j)$ with integer coordinates, $1 \le i \le n$, $1 \le j \le n$. It is required to colour each unit square in such a way that whenever $1 \le i < j \le n$ and $1 \le k < l \le n$, the three squares with centres at $(i, k), (j, k), (j, l)$ have distinct colours. What is the least possible number of colours needed?

**Solution:** We first show that at least $2n - 1$ colours are needed. Observe that squares with centres $(i, 1)$ must all have diffrent colours for $1 \le i \le n$; let us call them $C_1, C_2, \ldots, C_n$. Besides, the squares with centres $(n, j)$, for $2 \le j \le n$ must have different colours and these must be different from $C_1, C_2, \ldots, C_n$. Thus we need at least $n + (n - 1) = 2n - 1$ colours. The following diagram shows that $2n - 1$ colours will suffice.

| $n$ | $n+1$ | $n+2$ | ⋯ | | $2n-2$ | $2n-1$ |
|-----|-------|-------|---|---|--------|--------|
| $n-1$ | $n$ | $n+1$ | ⋯ | | $2n-3$ | $2n-2$ |
| ⋮ | ⋮ | ⋮ | | | ⋮ | ⋮ |
| $3$ | $4$ | $5$ | ⋯ | | $n+1$ | $n+2$ |
| $2$ | $3$ | $4$ | | | $n$ | $n+1$ |
| $1$ | $2$ | $3$ | ⋯ | | $n-1$ | $n$ |

5. Let $\Omega$ be a circle with a chord $AB$ which is not a diameter. Let $\Gamma_1$ be a circle on one side of $AB$ such that it is tangent to $AB$ at $C$ and internally tangent to $\Omega$ at $D$. Likewise, let $\Gamma_2$ be a circle on the other side of $AB$ such that it is tangent to $AB$ at $E$ and internally tangent to $\Omega$ at $F$. Suppose the line $DC$ intersects $\Omega$ at $X \ne D$ and the line $FE$ intersects $\Omega$ at $Y \ne F$. Prove that $XY$ is a diameter of $\Omega$.

**Solution:** Draw the tangent $PQ$ at $D$ such that $D$ is between $P$ and $Q$. Join $D$ to $A$, $B$ and $C$. Let $L = DA \cap \Gamma_1$ and $M = DB \cap \Gamma_1$. Join $C$ to $L$ and $M$. Observe that

$$\angle ADP = \angle LMD = \angle ABD. \tag{1}$$

Therefore $LM$ is parallel to $AB$ and hence $\angle LMC = \angle MCB$ (alternate angles). Again observe that

$$\angle ADC = \angle LDC = \angle LMC = \angle MCB = \angle MDC = \angle BDC. \tag{2}$$

Thus $CD$ bisects $\angle ADB$. Hence $X$ is the midpoint of the arc $AB$ not containing $D$. Similarly $Y$ is the midpoint of the arc $AB$ not containing $F$. Thus the arc $XY$ is half of the sum of two arcs that together constitute the circumference of $\Omega$ and hence it is a diameter.
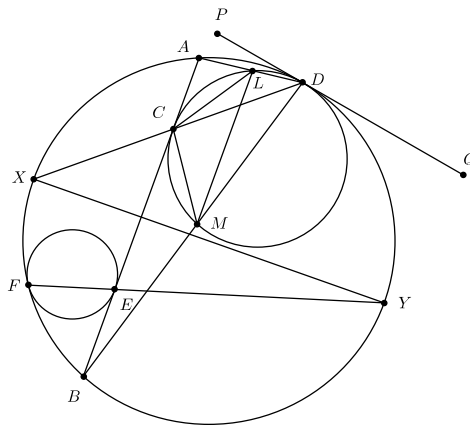


6. Let $x, y, z$ be real numbers, each greater than 1. Prove that

$$\frac{x+1}{y+1} + \frac{y+1}{z+1} + \frac{z+1}{x+1} \leq \frac{x-1}{y-1} + \frac{y-1}{z-1} + \frac{z-1}{x-1}.$$

**Solution:** We may assume that $x = \max\{x, y, z\}$. There are two cases: $x \geq y \geq z$ and $x \geq z \geq y$. We consider both these cases. The inequality is equivalent to

$$\left\{\frac{x-1}{y-1} - \frac{x+1}{y+1}\right\} + \left\{\frac{y-1}{z-1} - \frac{y+1}{z+1}\right\} + \left\{\frac{z-1}{x-1} - \frac{z+1}{x+1}\right\} \geq 0.$$

This is further equivalent to

$$\frac{x-y}{y^2-1} + \frac{y-z}{z^2-1} + \frac{z-x}{x^2-1} \geq 0.$$

Suppose $x \geq y \geq z$. We write

$$\frac{x-y}{y^2-1} + \frac{y-z}{z^2-1} + \frac{z-x}{x^2-1} = \frac{x-y}{y^2-1} + \frac{y-z}{z^2-1} + \frac{z-y+y-x}{x^2-1}.$$

This reduces to

$$(x-y)\frac{(x^2-y^2)}{(x^2-1)(y^2-1)} + (y-z)\frac{(x^2-z^2)}{(x^2-1)(z^2-1)}.$$

Since $x \geq y$ and $x \geq z$, this sum is nonnegative.

Suppose $x \geq z \geq y$. We write

$$\frac{x-y}{y^2-1} + \frac{y-z}{z^2-1} + \frac{z-x}{x^2-1} = \frac{x-z+z-y}{y^2-1} + \frac{y-z}{z^2-1} + \frac{z-x}{x^2-1}.$$

This reduces to

$$(x-z)\frac{(x^2-y^2)}{(x^2-1)(y^2-1)} + (z-y)\frac{(z^2-y^2)}{(y^2-1)(z^2-1)}.$$

Since $x \geq z$ and $z \geq y$, this sum is nonnegative.

Thus

$$\frac{x-y}{y^2-1} + \frac{y-z}{z^2-1} + \frac{z-x}{x^2-1} \geq 0$$

in both the cases. This completes the proof.

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